GEOMETRIC RELATIONS OF ABSOLUTE AND ESSENTIAL SPECTRA OF WAVE TRAINS

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Abstract. We analyze geometric relations of absolute and essential spectra for certain linear operators on the real line with periodic coefficients. These spectra correspond to accumulation sets of eigenvalues for increasing domain length under separated and periodic boundary conditions, respectively. The main result shows that critical isolated sets of essential spectrum contain absolute spectrum and yields an algorithm for its numerical computation. Linearizations of reaction diffusion systems in wave trains are used as an illustration, and we present a detailed numerical study of absolute and essential spectra for a wave train in the Schnakenberg model.

Key words. absolute spectrum; essential spectrum; isolated spectral sets; reaction diffusion systems

AMS subject classifications. 37L15; 34L16; 76E15; 47A10

1. Introduction.

We consider spectral properties of certain linear operators on the real line with periodic coefficients. Posed on an interval $(-L, L)$ with periodic boundary conditions, eigenvalues of these operators accumulate, as $L \to \infty$, on the *essential spectrum* of the operator posed on the real line. However, for generic separated boundary conditions eigenvalues accumulate on the so-called *absolute spectrum* introduced in [13]. In the presence of convection, absolute and essential spectra typically differ, and the absolute spectrum may be contained in the open left half plane, while the essential spectrum intersects the open right half plane. The absolute spectrum has been characterized in [13] via a complex dispersion relation of temporal and spatial modes. It was formulated for asymptotically constant coefficients, but all main results remain valid for the periodic case, cf. [13] p. 239.

In this article we analyze the relation of absolute and essential spectra for the case of periodic coefficients. Our main result, Theorem 4.5, shows that critical isolated sets of essential spectrum contain absolute spectrum. The proof provides an algorithm for the detection of this absolute spectrum, which we use for numerical computations in a specific example. A weaker version of this theorem is contained in [10] (Theorem 3.3).

The main application for the abstract framework are linearizations of parabolic partial differential equation (PDE), e.g. reaction diffusion systems, in traveling waves that are asymptotically periodic or that have an extended periodic region and are asymptotically constant in space. We refer to spatially periodic traveling waves as *wave trains*. In fact, absolute and essential spectra are determined by the spatially asymptotic states and we can restrict to wave trains for their study. We summarize the relevance of the absolute spectrum of traveling waves on large bounded domains as follows and refer to [12, 13] for details.

1. *Accumulation of eigenvalues:* On a bounded domain of length L only point spectrum occurs, but most of it accumulates as $L \to \infty$: For periodic boundary conditions at the essential spectrum, but for generic separated boundary conditions at the absolute spectrum.
2. *Absolute, remnant versus convective instabilities:* If the absolute spectrum is unstable, then perturbations grow pointwise or are convected in both direc-

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tions. Pointwise growth occurs when a branch point of the dispersion relation (Definition 3.1) in the absolute spectrum is unstable.

3. *Absolute instabilities are inherited:* If the profile of a traveling wave on the real line with constant asymptotics is close to another uniform steady state or wave train for a segment of length L, then, as $L \to \infty$, point spectrum accumulates at certain parts of the absolute spectrum of that state. An absolutely unstable state implies instability for all sufficiently large L.
4. *Linear spreading speeds:* The largest and smallest speeds of comoving frames in which the absolute spectrum is marginally stable yield linear predictions for spreading speeds of instabilities.

More formally, we consider linear operators on the real line as in [13], but explicitly assume periodicity of the coefficient matrix. These are a family $\mathcal{T}(\lambda)$, $\lambda \in \mathbb{C}$, where

$$\mathcal{T}(\lambda) : H^1(\mathbb{R}, \mathbb{C}^n) \to L^2(\mathbb{R}, \mathbb{C}^n), \quad u \mapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u$$

for the usual Sobolev space H^1 of L^2 functions with weak derivative in L^2. Here $A(\xi;\lambda) \in \mathbb{R}^{n\times n}$ is a matrix-valued function that satisfies the following.

HYPOTHESIS 1. *The matrices $A(\xi;\lambda)$ are smooth in $\xi \in \mathbb{R}$ and analytic in $\lambda \in \mathbb{C}$.*

1. *Periodic coefficients: there is $L > 0$ such that $A(\xi + L;\lambda) = A(\xi;\lambda)$ for all $\xi \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.*
2. *Well-posedness: There exists $\rho \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with real part $\Re(\lambda) \geq \rho$ the period map of the evolution of $\mathcal{T}(\lambda)v = 0$ has no Floquet exponent in $\mathrm{i}\mathbb{R}$.*

We use Hypothesis 1 as the abstract basis for studying absolute spectra and illustrate the results using reaction diffusion systems. In Lemma 3.3 we will show that Hypothesis 1 holds for the matrices arising in eigenvalue problems of wave trains in reaction diffusion systems in one space dimension.

Reaction diffusion systems consist of N 'species' $U = (U_1, \dots U_N) \in \mathbb{R}^N$ that are spatially coupled by diffusion, $D := \mathrm{diag}(d_1, \dots, d_N)$ with $d_j > 0$, and driven by pointwise reaction kinetics $F : \mathbb{R}^N \to \mathbb{R}^N$ in the form

$$U_t = DU_{xx} + F(U). \tag{1.1}$$

We assume this equation is posed on a function space X so that $D\partial_{xx}$ can be cast as a closed and densely defined operator, and the Nemitskij operator F_{N} derived from F satisfies $F_{\mathrm{N}} \in C^1(X, X)$. For example $X = BC^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{R}^N)$, domain of definition $\mathrm{dom}(D\partial_{xx}) = BC^2_{\mathrm{unif}}(\mathbb{R}, \mathbb{R}^N)$, and $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, cf. Chapter 2 in [17].

In a comoving frame with the variable $\xi = x - ct$ and speed c system (1.1) becomes

$$U_t = DU_{\xi\xi} + cU_\xi + F(U), \tag{1.2}$$

and we call t-independent solutions *traveling waves*. These solve the spatial ordinary differential equation (ODE)

$$DU_{\xi\xi} + cU_\xi + F(U) = 0, \tag{1.3}$$

and are called *wave trains* if $U(\xi + L) = U(\xi)$ for all ξ and some $L > 0$.

In case $d_j = 0$ for one or more j, spectra of linearizations of (1.3) about traveling waves with $c \neq 0$ are typically continuous as $d_j \to 0$, [10] Theorem 3.5.

This article is organized as follows. In §2 we define and mention the relevance of spectra for stability. Absolute and essential spectra are characterized in §3. In §4, we investigate the relative location of absolute and essential spectra, and prove the main result. We present numerical computations of (generalized) absolute and essential spectra for a specific example in §5.

2. Spectra and stability of wave trains. Following [13], we define the so-called essential spectrum of $\mathcal{T}$.

DEFINITION 2.1. *We say that $\lambda \in \mathbb{C}$ lies in the essential spectrum Σ_{ess} of $\mathcal{T}$, if $\mathcal{T}(\lambda)$ is not boundedly invertible. The essential spectrum of $\mathcal{T}$ is called (strictly) stable, if it lies in the (open) left half plane $\{\Re(\lambda) \leq 0\}$, and unstable if it intersects the open right half plane.*

As an illustration, consider a solution U^* to (1.3) as a steady state of (1.2). Its spectral stability is determined by the spectrum of the linearization of (1.2) about U^*, which is the linear operator

$$\mathcal{L} := D\partial_{\xi\xi} + c\partial_\xi + \partial_U F(U^*). \tag{2.1}$$

Generally, λ lies in the spectrum of $\mathcal{L}$, if the eigenvalue problem

$$\lambda V = \mathcal{L}V \tag{2.2}$$

has a bounded solution V. More precisely, we define the spectrum of U^* as follows.

DEFINITION 2.2. *The spectrum of $\mathcal{L}$, $\Sigma(\mathcal{L})$, is the set of $\lambda \in \mathbb{C}$ for which the operator $\mathcal{L} - \lambda$ is not boundedly invertible in X. The point spectrum of U^*, $\Sigma_{\text{pt}}(\mathcal{L})$, is the set of all $\lambda \in \Sigma(\mathcal{L})$ for which $\mathcal{L} - \lambda$ is a Fredholm operator with index zero. The essential spectrum of $\mathcal{L}$ is $\Sigma_{\text{ess}}(\mathcal{L}) := \Sigma(\mathcal{L}) \setminus \Sigma_{\text{pt}}(\mathcal{L})$. We call these spectra (strictly) stable respectively, if they lie in the (open) left half plane, and unstable if parts lie in the open right half plane.*

Definitions 2.1 and 2.2 are consistent in the sense that for a wave train U^* the spectrum of $\mathcal{L}$ equals that of $\mathcal{T}$ for the period L of U^*, see e.g. [6, 14]. In this case, the connection of $\mathcal{L}$ and $\mathcal{T}$ is as follows. We cast (2.2) as a linear non-autonomous first order ODE in $\mathbb{R}^{2N}$

$$\dot{v} = A_{\mathcal{L}}(\xi; \lambda)v, \tag{2.3}$$

corresponding to $\mathcal{T}(\lambda)v = 0$ and, according to (1.2), has the matrix

$$A_{\mathcal{L}}(\xi; \lambda) = \begin{pmatrix} 0 & \text{Id} \\ D^{-1}\partial_U F(U^*(\xi)) & D^{-1}c \end{pmatrix} - \lambda \begin{pmatrix} 0 & 0 \\ D^{-1} & 0 \end{pmatrix}.$$

For any traveling wave U^* of (1.1), the derivative $\partial_\xi U^*$ solves the eigenvalue problem (2.2), and zero lies in the spectrum of U^* ('Goldstone mode'). Therefore, we cannot expect asymptotic stability on unbounded or periodic domains. Nevertheless, two types of weakened nonlinear stability can be concluded from spectral properties: nonlinear stability with asymptotic phase and diffusive stability. The former follows from a simple zero eigenvalue and a spectral gap to the strictly stable rest of the spectrum: a perturbed wave converges exponentially to a selected translate of the original wave, see e.g. [8] chapter 5.1. However, the required spectral gap does not occur for wave trains on the real line, where the essential spectrum comes in curves. Instead, wave trains can be stable in the sense that perturbations 'diffuse' self-similarly over the wave train [16].

On the other hand, unstable spectrum causes nonlinear instability for any traveling wave [8].

3. Essential and absolute spectra. Let $\Phi_\lambda(\xi,\zeta)$ denote the evolution of (2.3). It has a Floquet representation with L-periodic matrix $S_\lambda(\xi)$, $S_\lambda(0)=\mathrm{Id}$, of the form

$$\Phi_\lambda(\xi,0) = S_\lambda(\xi)\mathrm{e}^{R(\lambda)\xi}.$$

DEFINITION 3.1.

- *The complex dispersion relation $d:\mathbb{C}^2\to\mathbb{C}$ of $\mathcal{T}$ is defined by*

$$d(\lambda,\nu) = \det(R(\lambda)-\nu) = 0,$$

- *a branch point $\lambda\in\mathbb{C}$ at $\nu\in\mathbb{C}$ is such that $d(\lambda,\nu)=\partial_\nu d(\lambda,\nu)=0$,*
- *the spatial Morse index $i(\lambda)$ counts the center unstable dimensions of $R(\lambda)$,*
- *eigenvalues $\nu(\lambda)$ of $R(\lambda)$ are called spatial Floquet exponents.*

Under Hypothesis 1, the matrix $R(\lambda)$ is analytic in λ, cf. e.g. [3]. Note that spatial Floquet exponents are unique modulo $2\pi\mathrm{i}$.

It is well known that the essential spectrum of $\mathcal{T}$ consists of those λ for which a spatial Floquet exponent is purely imaginary [6], that is,

$$\lambda\in\Sigma_{\mathrm{ess}} \Leftrightarrow \exists k\in\mathbb{R}\,:\, d(\lambda,\mathrm{i}k)=0. \tag{3.1}$$

Hence the spatial Morse index $i(\lambda)$ is constant in connected components of $\mathbb{C}\setminus\Sigma_{\mathrm{ess}}$ and whenever $\partial_\lambda d(\lambda,\nu)\partial_\nu d(\lambda,\nu)\neq 0$ in the essential spectrum, the implicit function theorem locally yields a unique smooth curve of Σ_{ess}. We expect singularities to occur either on a discrete set, or that $d(\lambda,\nu)$ has a multiple factor; note that $d(\lambda,\nu)$ contains the term $R(\lambda)^n$ so that the Weierstrass Preparation Theorem applies, cf. [9], which reduces $d(\lambda,\nu)=0$ locally to roots of a polynomial with analytic coefficients in ν.

Concerning the global topology of Σ_{ess}, for constant matrix $A(\xi;\lambda)=A(\lambda)$ the essential spectrum is a connected set of curves in the closed complex plane, cf. e.g. [10]. However, for the present case with periodic matrix $A(\xi;\lambda)$, it can consist of several connected components. For instance, the essential spectrum of wave trains in reaction diffusion systems that are sufficiently close to a pulse (constant spatial asymptotics) decomposes in a certain way [7, 14], see also Corollary 4.4 and §5 below.

Hypothesis 1 implies that $i(\lambda)$ is constant for all $\Re(\lambda)\geq\rho$. The absolute spectrum, introduced first in [13], is defined in terms of this constant as follows.

DEFINITION 3.2. *The generalized absolute spectrum of $\mathcal{T}$ with Morse index j is*

$$\Sigma^j_{\mathrm{abs}} := \{\lambda\in\mathbb{C}\mid \Re(\nu_j)=\Re(\nu_{j+1})\},$$

where $\Re(\nu_1)\geq\Re(\nu_2)\geq\ldots\geq\Re(\nu_n)$ counted with multiplicity. The generalized absolute spectrum, Σ^*_{abs}, is the union of Σ^j_{abs} for $j=1,\ldots n-1$.

Finally, the absolute spectrum for $\mathcal{T}$ satisfying Hypothesis 1 is defined as

$$\Sigma_{\mathrm{abs}} := \Sigma^{i_\infty}_{\mathrm{abs}}\,,$$

where i_∞ is the constant number of Floquet exponents with positive real part given by the well-posedness in Hypothesis 1 for $\Re(\lambda)\geq\rho$.

To illustrate this definition we plot sample configurations of spatial Floquet exponents ν in Figure 3.1. By the implicit function theorem Σ^*_{abs} consists of smooth curves away from singularities. The additional requirement of a certain Morse index for Σ^j_{abs} typically causes corners in Σ^j_{abs} at points where smooth curves in Σ^*_{abs} cross

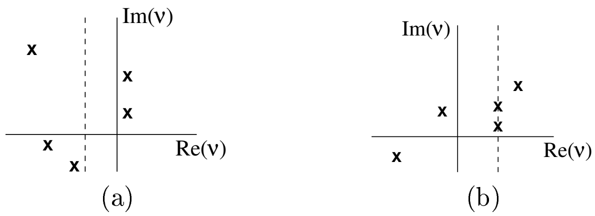


Fig. 3.1: We plot sample configurations of 5 Floquet exponents and $i_\infty = 2$ for some λ: (a) λ not in the absolute spectrum, (b) λ in the absolute spectrum.

transversely. Therefore, computing $\Sigma^*_{\rm abs}$ is a natural first step for the computation of the absolute spectrum, cf. [11]. Similar to the essential spectrum, the absolute spectrum is connected for constant coefficients, cf. [11], but is e.g. not connected for wave trains close to a pulse, see Corollary 4.4.

We next show that the abstract framework indeed applies to wave trains in reaction diffusion systems with $i_\infty = N$. In §2, we identified the spectrum of the linearization of a reaction diffusion system about a wave train as the essential spectrum of $\mathcal{T}$ for a certain matrix $A = A_{\mathcal{L}}$. The arising operator $\mathcal{T}$ satisfies Hypothesis 1 as follows.

Lemma 3.3. *The family of operators $\mathcal{T}_{\mathcal{L}}(\lambda)$ derived from (1.1) for a wave train U^* as in (2.3) satisfies Hypothesis 1. More precisely, there is $R \in \mathbb{R}$ such that for $\Re(\lambda) > R$ the spatial Morse index is $i(\lambda) = N$, and for all spatial Floquet exponents ν corresponding to $\mathcal{T}_{\mathcal{L}}(\lambda) = 0$ it holds that $|\Re(\nu)| \to \infty$ as $\Re(\lambda) \to \infty$.*

Proof. Consider $D\ddot{V} + c\dot{V} + \partial_U F(U_*(\xi))V - \lambda V = 0$. Rescaling to fast 'spatial time' $\xi = \epsilon\zeta$, $' = \frac{\rm d}{{\rm d}\zeta}$ and $\lambda = \tilde\lambda/\epsilon^2$ we obtain

$$\begin{aligned} DV''/\epsilon^2 + cV'/\epsilon + \partial_U F(U_*(\epsilon\zeta))V - \tilde\lambda/\epsilon^2 V = 0\\ (\epsilon \neq 0) \Leftrightarrow DV'' + \epsilon cV' + \epsilon^2\partial_U F(U_*(\epsilon\zeta))V - \tilde\lambda V = 0 \end{aligned}$$

For $\epsilon = 0$ the latter 'fast system' is $DV'' = \tilde\lambda V$ and has dispersion relation $\Pi_{j=1}^N(d_j\nu^2 - \tilde\lambda) = 0$, $d_j > 0$. For $\Re(\tilde\lambda) > 0$ its spatial Morse index is N and there are no spatial Floquet exponents on the imaginary axis. Therefore, for all $\Re(\tilde\lambda) > 0$, the fast system has exponential dichotomies on $\mathbb{R}^\pm$ with Morse index N and exponential rate $\min\{\sqrt{\tilde\lambda/d_j}\}$. By roughness of exponential dichotomies, e.g. [4], the spatial Morse index, and the exponential rate (with $o(1)_{\epsilon\to 0}$ adjustment) persist for small bounded perturbation, i.e. for $\epsilon > 0$ sufficiently small.

In particular, the exponential rates of the dichotomies in the normal 'time' ξ for $0 < \epsilon \ll 1$ are bounded from below by

$$\frac{1}{2}\min\left\{\Re\left(\epsilon^{-1}\sqrt{\frac{\tilde\lambda}{d_j}}\right)\right\} = \epsilon^{-1}\frac{1}{2}\min\left\{\Re\left(\sqrt{\frac{\tilde\lambda}{d_j}}\right)\right\}.$$

Since the precise exponential rate is given by the spatial Floquet exponent ν that is closest to the imaginary axis, the lower bound implies that all spatial Floquet exponents have unbounded real part as $\Re(\lambda) \to \infty$, i.e. $\epsilon \to 0$ for fixed $\tilde\lambda$.

Periodicity and smoothness in ξ are evident, which completes Hypothesis 1. ∎

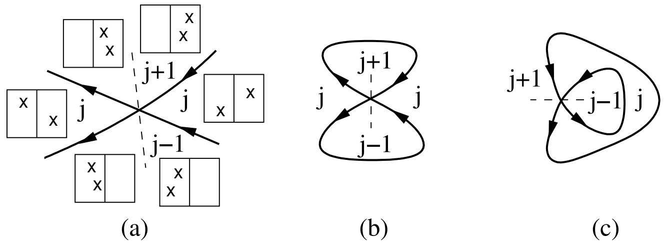


Fig. 4.1: We sketch intersection points in $\Sigma_{\rm abs}^j$ of two curves of $\Sigma_{\rm ess}$ (solid lines) oriented by increasing k, and show the Morse indices in the neighboring regions. Dashed lines indicate the curve of $\Sigma_{\rm abs}^j$ that typically crosses the intersection point and insets in (a) the distribution of ν_j and ν_{j+1} in $\mathbb{C}$ (vertical line is $\mathrm{i}\mathbb{R}$). Plots (b) and (c) illustrate the case of isola.

4. Relative location of absolute and essential spectra. A straightforward constraint for the location of absolute spectrum is that it must lie on or to the left of the essential spectrum in the complex plane. To be definite we formulate the following lemma.

LEMMA 4.1. *Under Hypothesis 1, the connected component $\Omega_\infty \subset \mathbb{C} \setminus \Sigma_{\rm ess}$ that contains an unbounded interval of $\mathbb{R}_+$ is well defined. Any curve that connects $\Sigma_{\rm abs}$ and Ω_∞ intersects $\Sigma_{\rm ess}$.*

Proof. By (3.1) and Hypothesis 1 we have $\{\lambda \mid \Re(\lambda) \geq \rho\} \cap \Sigma_{\rm ess} = \emptyset$, hence $\Omega_\infty \supset \{\lambda \mid \Re(\lambda) \geq \rho\}$. Therefore, Ω_∞ is well defined and $i(\lambda) = i_\infty$ for any $\lambda \in \Omega_\infty$. This is not possible for $\lambda \in \Sigma_{\rm abs}$, because there $i(\lambda) \geq i_\infty + 1$ or $i(\lambda) \leq i_\infty - 1$. Since $\partial\Omega_\infty \subset \Sigma_{\rm ess}$, the intersection claim follows. ◻

If the spatial Morse index changes at some $\lambda_0 \in \Sigma_{\rm ess}$ while moving *within* the essential spectrum, then $\lambda_0 \in \Sigma_{\rm abs}^*$, because two spatial Floquet exponents have the same real part, namely zero. Typically, λ_0 is an intersection of two curves in the essential spectrum, which gives some more information about the location of absolute spectrum, see also Figure 4.1 and §5.

LEMMA 4.2. *Suppose that two curves in $\Sigma_{\rm ess}$ intersect at λ_0, but $\partial_\lambda d(\lambda_0, \nu) \neq 0$ for all spatial Floquet exponents $\nu \in \mathrm{i}\mathbb{R}$. Then λ_0 lies in the generalized absolute spectra with Morse indices $j_- + 1, \ldots, j_+ - 1$, where $j_- = \liminf_{\epsilon\to 0}\{i(\lambda) \mid \epsilon = |\lambda - \lambda_0|\}$ and $j_+ = \limsup_{\epsilon\to 0}\{i(\lambda) \mid \epsilon = |\lambda - \lambda_0|\}$.*

Assume further that these curves intersect transversely and are $\lambda_\iota(\mathrm{i}k)$ with $\lambda_0 = \lambda_\iota(\mathrm{i}k_\iota)$, $\frac{\mathrm{d}}{\mathrm{d}k}|_{k=k_\iota}\lambda_\iota(\mathrm{i}k) \neq 0$ for $\iota = 1, 2$, and that $\Re(\nu) \neq 0$ for any other solution to $d(\lambda_0, \nu) = 0$. Then there exists a locally unique curve $C \subset \Sigma_{\rm abs}^j$, $j = j_+ - 1 = j_- + 1$, crossing at λ_0 from the region where $i(\lambda) = j_-$ into the region where $i(\lambda) = j_+$.

Proof. Such an intersection point lies in the generalized absolute spectrum, because either two different purely imaginary spatial Floquet exponents have the same real part, or it is a branch point. Since the Morse index counts the number of unstable and purely imaginary Floquet exponents, it follows from the ordering of spatial Floquet exponents at λ_0 by decreasing real parts that the largest and smallest indices ℓ so that $\Re(\nu_\ell) = \Re(\nu_{\ell+1})$ are $j_+ - 1$ and $j_- + 1$.

The regularity assumptions imply that precisely two curves of essential spectrum intersect transversely so that $\mathbb{C} \setminus \Sigma_{\rm ess}$ is divided into four sectors near λ_0 and $i(\lambda) = j$ in two opposing sectors, and $i(\lambda) = j \pm 1$ in the other two, respectively. By the implicit function theorem there exists a locally unique curve $C \subset \Sigma_{\rm abs}^j$, which crosses

λ_0 transversely. Finally, note that C cannot intersect the sector with index j or $\Sigma_{\rm ess}$ near λ_0. □

REMARK 1. *In a weighted space X_η with norm $\sup_{\xi\in\mathbb{R}} e^{\eta\xi}|u(\xi)|$ the essential spectrum is $\Sigma_{\rm ess}^\eta = \{\lambda \mid d(\lambda, ik-\eta) = 0\}$. For reaction diffusion systems (1.1), the asymptotics of $\Re(\nu)$ stated in Lemma 3.3 imply that for any weight η there is a connected component $\Omega_\infty^\eta \subset \mathbb{C}$ with the same constant Morse index i_∞ and $\Omega_\infty^0 = \Omega_\infty$. Therefore, constraints on the relative location of essential and absolute spectra hold true for the absolute spectrum and $\Sigma_{\rm ess}^\eta$ as well. We expect that a singularity in $\Sigma_{\rm ess}$ can be removed by changing η, hence regularity assumptions, as in Lemma 4.2, should not be very restrictive. In fact, it can be more efficient to numerically continue the essential spectrum in an exponential weight and thereby infer location of absolute spectrum, than to compute the absolute spectrum itself, e.g. [11].*

Our main result implies absolute spectrum within *isola* in $\Sigma_{\rm ess}$, i.e. closed bounded curves. As a motivation, consider the aforementioned wave trains near a pulse on a large bounded domain. Unstable essential spectrum might 'only' cause a convective instability that is not seen under separated boundary conditions. However, an instability of the point spectrum of a pulse should heuristically destabilize nearby wave trains. Indeed, point spectrum in Ω_∞ generates isola of essential spectrum for nearby wave trains [7, 14] that contain absolute spectrum on account of Corollary 4.4 below. Therefore, instabilities of point spectrum of pulses are inherited to nearby wave trains, at least on sufficiently large domains.

A Jordan curve in the complex plane is a closed, bounded curve $\gamma \subset \mathbb{C}$ without self-intersections. It is well known that such a curve divides the complex plane into an 'exterior' set, which is an unbounded connected component of $\mathbb{C}$, and its complementary 'interior' set, $\mathrm{int}(\gamma)$, which is a bounded connected component of $\mathbb{C}$. More generally, for an isola $\gamma \subset \mathbb{C}$, i.e. γ is a closed bounded curve, we define

$$\begin{aligned} M(\gamma) &:= \bigcap \{\mathrm{int}(\gamma') \mid \gamma' \text{ is a Jordan curve and } \gamma \subset \mathrm{int}(\gamma')\} \\ \mathrm{int}(\gamma) &:= M(\gamma) \setminus \partial M(\gamma). \end{aligned}$$

Note that $\mathrm{int}(\gamma)$ may consist of several connected components, and may also be empty, e.g., if γ is an interval.

We say $\lambda_0 \in \Sigma_{\rm ess}$ is a *regular point*, if $\partial_\lambda d(\lambda_0,\nu)\partial_\nu d(\lambda_0,\nu) \neq 0$ for all $\nu \in i\mathbb{R}$ with $d(\lambda_0,\nu) = 0$.

PROPOSITION 4.3. *Assume Hypothesis 1 and that an isola in $\Sigma_{\rm ess}$ contains a non-empty connected component $K \subset \mathbb{C}\setminus\Sigma_{\rm ess}$ with a regular point on ∂K and $i(\lambda)$ is constant for $\lambda \in \partial K$. Then $\bar{K} \cap \Sigma_{\rm abs}^j \neq \emptyset$, where $j = i(\partial K)$ if $i(K) < i(\partial K)$, and $j = i(\partial K) - 1$ otherwise.*

The proof of this proposition follows below. Note that the spatial Morse indices $i(\partial K)$ and $i(\partial K) - 1$ distinguish increasing or decreasing Morse index when entering K. For an isola at the origin this is (typically) determined by the group velocity $-\mathrm{d}\lambda/\mathrm{d}\nu|_{\nu=0}$.

Using the Cauchy-Riemann equations, Proposition 4.3 yields [10] Theorem 3.3. Together with [14] Theorem 2.1 it directly implies the following corollary about wave trains near a pulse, which we formulate for reaction diffusion systems (1.1). Here the pulse is required to be a generic homoclinic orbit of (1.3) in the sense that the underlying parameter provides a transverse unfolding and the variational equation

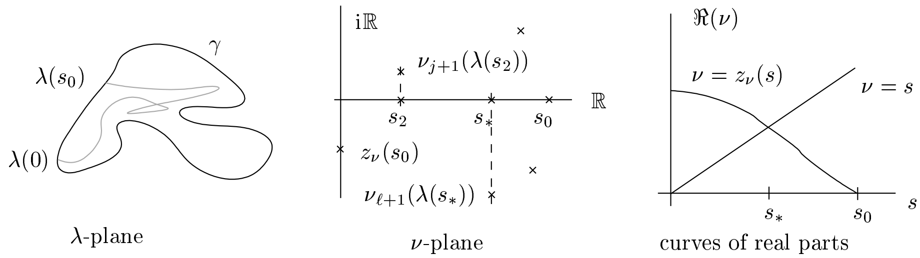


Fig. 4.2: Configuration of spatial Floquet exponents in the proof of Proposition 4.3.

has a unique bounded solution (up to constant multiples), see [14].

COROLLARY 4.4. *Suppose U_L^* is a family of wave trains of (1.1) with the following properties. As solutions to (1.3) we have $U_L^* \to U_\infty^*$, as $L \to \infty$, which is a generic homoclinic orbit to an equilibrium (i.e. it satisfies the assumptions of [14] Theorem 5.1). For any λ_* in the point spectrum of U_∞^* there are positive constants L_*, C and κ, such that for any $L \geq L_*$ there is an isola $\gamma_L \subset \Sigma_{\text{ess}}(U_L^*)$ with $\lambda_* \in \overline{\text{int}(\gamma_L)}$ and $\text{diam}(\gamma_L) \leq C\text{e}^{-\kappa L}$. Moreover, $\text{int}(\gamma_L) \cap \Sigma_{\text{abs}}^*(U_L^*) \neq \emptyset$ and $\gamma_L \subset \partial\Omega_\infty(U_L^*)$ implies $\text{int}(\gamma_L) \cap \Sigma_{\text{abs}}(U_L^*) \neq \emptyset$.*

REMARK 2. *It appears natural that isola in Ω_∞ contain branch points (Definition 3.1) in the absolute spectrum. This would be meaningful, because such points distinguish remnant and absolute instabilities. However, the problem is the Morse index: Having located a branch point by a homotopy, it seems difficult to exclude the possibility of an index change during this homotopy. Indeed absolute spectra without branch points can occur, see [11]§5.1.*

We combine the most relevant parts of Lemma 4.2 for isola and Proposition 4.3 in the following theorem; recall the definition of $j_\pm$ in Lemma 4.2.

THEOREM 4.5. *Assume Hypothesis 1 and that Σ_{ess} contains an isola γ so that an open neighborhood V of its interior satisfies $i(V \setminus \overline{\text{int}(\gamma)}) \equiv j$. Suppose one of the following.*

(i) $i(\text{int}(\gamma)) \equiv j+1$ or $i(\text{int}(\gamma)) \equiv j-1$, and $i(\lambda)$ is constant for $\lambda \in \gamma$.
(ii) γ self intersects at a regular point where $j_- + 1 \leq j \leq j_+ - 1$,
(iii) $\text{int}(\gamma) = \emptyset$ and γ contains a regular point.

Then $\overline{\text{int}(\gamma)} \cap \Sigma_{\text{abs}}^j \neq \emptyset$, in particular $\gamma \subset \partial\Omega_\infty$ implies γ contains absolute spectrum.

Note that assumption (ii) is satisfied for an exterior loop that is connected to the rest of the curve as in a figure eight shape, see Figure 4.1(b) and Lemma 4.2.

Proof. Case (i) follows directly from Proposition 4.3 and case (ii) immediately from Lemma 4.2.

At a regular point $\lambda_0 \in \gamma$ in case (iii), the Cauchy-Riemann equations imply that a Floquet exponent crosses the imaginary axis as λ crosses λ_0. By assumption, the Morse index cannot change when λ has crossed λ_0, hence a second Floquet exponent crosses the imaginary axis in the opposite direction. Therefore, two spatial Floquet exponents are purely imaginary at λ_0, and their indices in the ordering of decreasing real parts are necessarily j and $j+1$, so $\gamma \cap \Sigma_{\text{abs}}^j \neq \emptyset$.

Lastly, note that $\gamma \subset \partial\Omega_\infty$ yields $j = i_\infty$. □

Proof of Proposition 4.3.

Definition 4.6. *We say that a bounded curve $\lambda(s) \in \mathbb{C}$, $s \in [0, s_0]$, $s_0 > 0$, is Σ-connecting, if there exists a curve $\nu(s) \in \mathbb{C}$, $s \in [0, s_0]$, such that*

- *$\lambda(0) \in \Sigma_{\rm ess}$ is a regular point,*
- *$d(\lambda(s), \nu(s)) = 0$ for all $s \in [0, s_0]$,*
- *either $\partial_\nu d(\lambda(s_0), \nu(s_0)) = 0$ or $\lambda(s_0) \in \Sigma_{\rm ess}$, $\nu(s_0) \notin \mathrm{i}\mathbb{R}$,*
- *$i(\lambda(0)) = i(\lambda(s_0))$ in case $\nu(s_0) \in \mathrm{i}\mathbb{R}$,*
- *for $s \in (0, s_0)$ any solution ν to $d(\lambda(s), \nu) = 0$ satisfies $\Re(\nu) \neq 0$.*

We prove Proposition 4.3 in two steps. Firstly, we show that a Σ-connecting curve emanates from any regular point on ∂K. More precisely, we show that $\nu(s) = \mathrm{i}k_0 + s$ or $\nu(s) = \mathrm{i}k_0 - s$ can be used for certain k_0, which corresponds to continuing the essential spectrum in an exponentially weighted space as in Remark 1. Secondly, we prove that a Σ-connecting curve generally implies presence of absolute spectrum with a certain Morse index.

To illustrate the approach, consider the scalar constant coefficient equation $u_{xx} = cu_x$. It has the dispersion relation $d(\lambda, \nu) = \nu^2 - c\nu - \lambda = 0$, hence $\Sigma_{\rm ess} = \{c\mathrm{i}k - k^2 | k \in \mathbb{R}\}$ and $\Sigma_{\rm abs} = (-\infty, -c^2/4]$. In this case, for fixed $k \in \mathbb{R}$, the roots of $d(\lambda, s + \mathrm{i}k) = 0$ form the Σ-connecting curve $\lambda(s) = s^2 - k^2 - cs + \mathrm{i}k(2s - c)$ with $\nu(s) = s$ and appropriately chosen s_0. This curve indeed intersects the absolute spectrum at $s = \frac{c}{2}$, where $\lambda(\frac{c}{2}) = -c^2/4 - k^2 \in \Sigma_{\rm abs}$. In fact, $d(\lambda, \mathrm{i}k + \frac{c}{2}) = 0$ gives the entire absolute spectrum.

Lemma 4.7. *Assume the hypotheses of Proposition 4.3. For any regular $\lambda_0 \in \partial K$ there exists a Σ-connecting curve with $\lambda(s) \in \overline{\mathrm{int}(K)}$ for $s \in [0, s_0]$ with $\lambda(0) = \lambda_0$.*

Proof. The implicit function theorem provides $s_0 > 0$ and a locally unique curve $\lambda(s)$ for $|s| \in [0, s_0)$ with $\lambda(0) = \lambda_0$ and $d(\lambda(s), s + \mathrm{i}k_0) = 0$. Note that the Morse index $i(\lambda)$ changes by one at λ_0, but remains constant in K, because $K \cap \Sigma_{\rm ess} = \emptyset$.

Assume first that $\lambda(s) \in K$ for $s > 0$. In case $\partial_\lambda d(\lambda_0, \mathrm{i}k_0 + s_0) = 0$, we can use Rouché's theorem, cf. e.g. [1], to continue $\lambda(s)$ to larger values of s. Indeed, if roots could not be chosen continuously for increasing s, then the number of roots of $d(\cdot, \mathrm{i}k_0 + s_0)$ and $d(\cdot, \mathrm{i}k_0 + s_0 + \epsilon)$ would differ in a small neighborhood of λ_0 for any sufficiently small $\epsilon > 0$, which contradicts Rouché's theorem since $d(\cdot, \cdot)$ is smooth.

Using $K \cap \Sigma_{\rm ess} = \emptyset$ we can choose $s_0 > 0$ minimal, so that either $\partial_\nu d(\lambda(s_0), s_0 + \mathrm{i}k_0) = 0$ or $\lambda(s_0) \in \partial K$. The assumption of constant Morse index on ∂K implies $i(\lambda(0)) = i(\lambda(s_0))$.

In case or $\lambda(s) \in K$ for $s < 0$ the same holds for smaller, maximally chosen s_0. □

Lemma 4.8. *Assume Hypothesis 1 and that there exists a Σ-connecting curve. Then there exists $0 < s_* \leq s_0$ such that $\lambda(s_*) \in \Sigma^j_{\rm abs}$, where $j = i(\lambda(0))$ if $i(\lambda(s_0/2)) < i(\lambda(0))$, and $j = i(\lambda(0)) - 1$ otherwise.*

Proof. By definition of the spatial Morse index and Σ-connecting curves, $i(\lambda(s))$ is either constant for $[0, s_0]$ or $i(\lambda(0)) > i(\lambda(s))$ for $s \in (0, s_0)$. By regularity at λ_0, the definition of j covers all cases and we focus on $i(\lambda(0)) = j + 1$, i.e. $\Re(\nu(s)) > 0$ for $s \in (0, s_0)$, and comment on the case $i(\lambda(0)) = j$. The idea of proof with $\nu(s) = \mathrm{i}k_0 + s$ is sketched in Figure 4.2.

Step 1: We show that whenever the Σ-connecting curve contains generalized absolute spectrum involving $\nu(s_2)$ for some $s_2 \in [0, s_0]$, then it also contains absolute spectrum. We thus assume that $\lambda(s_2) \in \Sigma^\ell_{\rm abs}$ and $\nu_\ell = \nu(s_2)$ in the ordering of decreasing real parts. Since $i(\lambda(s)) \equiv j + 1$ for $s \in (0, s_0)$, no Floquet exponent crosses the imaginary axis and $\ell \in \{1, \ldots, j + 1\}$. The unique spatial Floquet exponents on the imaginary axis at $\lambda(0)$ is $\nu(0)$, hence $\nu(0) = \nu_{j+1}$. Since Floquet exponents are continuous along $\lambda(s)$, there is $s_* \in [0, s_2]$ such that $\lambda(s_*) \in \Sigma^j_{\rm abs}$, see Figure 4.2(b) for an illustration.

(*Case* $i(\gamma) = j$*:* the real parts of two stable Floquet exponents coincide, and $\nu(0) = \nu_j$ in the ordering, so $\ell \in \{j, \dots, n\}$. Again we infer $\ell = j$ at some point.)

Step 2: We establish generalized absolute spectrum in the Σ-connecting curve and apply the first step. In particular, the first step applies if $\lambda(s_0)$ is a branch point at $\nu(s_0)$, so that we may assume this is not the case. Therefore, $\lambda(s_0) \in \Sigma_{\text{ess}}$ and there is a solution $z_\nu(s_0) \in \mathrm{i}\mathbb{R}$ to $d(\lambda(s_0), \cdot) = 0$. By assumption, the Morse index is constant on ∂K and thus, if $\lambda(s_0)$ is a branch point at $z_\nu(s_0)$, then it lies in Σ^j_{abs}. Otherwise, Rouché's theorem (and typically the implicit function theorem) yields a curve $z_\nu(s) \in \mathbb{C}$ for $s \leq s_0$ near s_0 solving $d(\lambda(s), z_\nu(s)) = 0$. As in the proof of Lemma 4.7, we can choose maximal $0 \leq s_1 \leq s_0$ so that either $\lambda(s_1)$ is a branch point at $z_\nu(s_1)$, or $s_1 = 0$. Since the above first step applies analogously to $\lambda(s) \in \Sigma^*_{\text{abs}}$ involving $z_\nu(s)$, and a branch point lies in this set, we next assume $s_1 = 0$.

(*Case* $i(\lambda(0)) = j$*:* we can choose $s_1 \geq s_0$ and $z_\nu(s_0)$ so that $\Re(z_\nu(s)) < 0$.)

The regularity of $\lambda(0)$ implies that $\mathrm{i}k_0$ is the locally unique solution to $d(\lambda(0), \nu) = 0$ for $\nu \in \mathrm{i}\mathbb{R}$, and $\lambda(0)$ that of $d(\lambda, \mathrm{i}k_0) = 0$ for $\lambda \in \mathbb{C}$. Therefore, $\Re(z_\nu(0)) > 0$ and by construction $\Re(z_\nu(s_0)) = 0$, $\Re(\nu(s_0)) > 0$, $\Re(\nu(0)) = 0$. Hence, by continuity, $\Re(z_\nu(s_2)) = \Re(\nu(s_2))$ at some $s_2 \in (0, s_0)$, so that $\lambda(s_2)$ lies in the generalized absolute spectrum at $\nu(s_2)$. Applying the first step concludes the proof. □

Lemma 4.7 and Lemma 4.8 together prove Proposition 4.3. Note that Lemma 4.8 applies to any Σ-connecting curve, including ones that emanate from unbounded curves of essential spectrum.

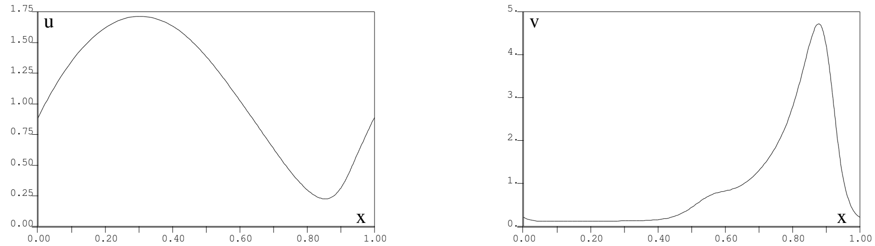
REMARK 3. *The constructive nature of the proof of Proposition 4.3 yields an algorithm for the location generalized absolute spectrum in an isola by numerical continuation, see also [11]: Given an isola γ, pick $\lambda \in \gamma$ and continue the solution $\lambda(s)$ to $d(\lambda, s + \mathrm{i}k_0) = 0$ in s such that it moves into* $\mathrm{int}(\gamma)$*. We expect that $\partial_\lambda d(\lambda, \nu) \neq 0$ along this curve, and either a branch point occurs, or $\lambda(s)$ intersects γ for some $|s_0| > 0$. In the latter case, compute $d(\lambda(s_0), \mathrm{i}k_1) = 0$ with $\mathrm{i}k_0 \neq \mathrm{i}k_1$ and continue the two Floquet exponents back along the curve $\lambda(s)$ until either a branch point occurs or the difference in real parts is zero.*

Thus, a point Σ^ℓ_{abs} has been located (unless a double root with respect to λ occurred), and in practice often $\ell = j$, see §5 for examples. To handle the case $\ell \neq j$, we can extend the algorithm as follows. Pick $\lambda \in \gamma$ and find all n solutions to $d(\lambda, \nu) = 0$, in particular there is $\nu = \mathrm{i}k$. Then continue all solutions simultaneously as above in s. The theorem guarantees (unless a double root with respect to λ occurred) that there is $0 \leq |s| \leq |s_0|$ such that $\Re(\nu_j) = \Re(\nu_{j+1})$. We refer to [11] for algorithms to find all initial ν, and for notes on the implementation in the software AUTO *[5].*

5. An example. We present computations of essential and generalized absolute spectra for a wave train that occurs in the Schnakenberg model [15] using the numerical methods described in Remark 3 and [11] with the software AUTO [5]. Both absolute and essential spectrum of this wave train are unstable. We aim to illustrate the possible structures of (generalized) absolute and essential spectra of wave trains, and the use of Theorem 4.5 in calculating them.

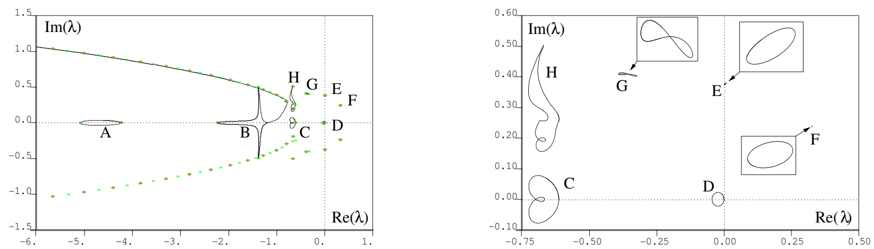
We use the Schnakenberg model in the form

$$
\begin{aligned}
u_t &= D_u u_{xx} + 0.9 - uv^2 \\
v_t &= D_v v_{xx} + 0.1 + uv^2 - v,
\end{aligned}
$$



(a) (b)

Fig. 5.1: (a) u-component of the wave train U^*. (b) v-component.



(a) (b)

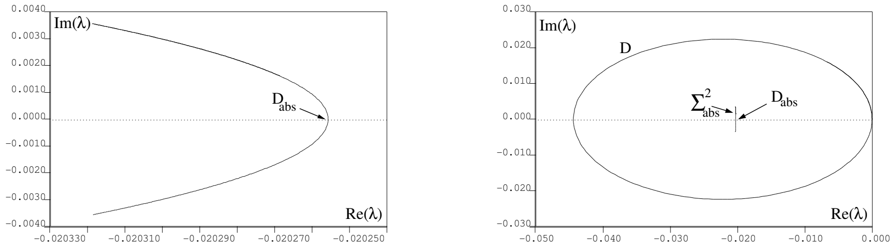
Fig. 5.2: (a) Parts of essential spectrum of the wave train computed with AUTO (black curves) and eigenvalues computed with finite differences on $[0, 3]$ (stars) and $[0, 6]$ (crosses). Letters denote isola referred to in the text. (b) magnifies part of (a) and insets magnify the indicated isola.

and consider a wave train U^* of period $L = 3$ for the parameters $D_u \approx 0.45$, $D_v \approx 0.0045$ and velocity $c \approx 0.029$. The profiles of U^* are plotted in Figure 5.1.

This wave train can be located as follows. For $D_u \equiv 1$ the unique equilibrium $u = v = 0$ undergoes a Turing bifurcation at $D_v \approx 0.12$. We continue the bifurcating Turing pattern with period $L = 3$ to $D_u = 1$, $D_v = 0.01$ and then fix the ratio $D_v/D_u \equiv 0.01$. A bifurcation to a traveling wave train occurs as D_u decreases and we consider the solution U^* on this branch at $D_u \approx 0.45$ in the comoving frame with its speed $c \approx 0.029$.

Recall that essential and generalized absolute spectra are determined by solutions of the (complex) dispersion relation (3.1). All $(\lambda, \nu) \in \mathbb{C}^2$ in the following are such solutions, viewed as local functions $\lambda(\nu)$ or $\nu(\lambda)$. The essential spectrum are all branches of $\lambda(\mathrm{i}k)$ and the generalized absolute spectrum all λ for which two branches $\nu(\lambda)$ have the same real part. For this model there are four Floquet exponents ν at each λ, i.e. four branches of $\nu(\lambda)$, and these connect at branch points (Definition 3.1). To locate λ in the absolute spectrum (here $i_\infty = N = 2$) the spatial Morse index $i(\lambda)$ is needed, and therefore all four Floquet exponents at λ.

The search for essential spectrum was guided by predictions from eigenvalues of first order finite difference approximations. Figure 5.2 plots the critical parts of the



(a) (b)

Fig. 5.3: (a) Curve of absolute spectrum inside isola D. (b) This curve together with isola D. For the point D_{abs} see also Figure 5.6.

essential spectrum computed with AUTO [5] as well as eigenvalues computed via finite difference discretization with LAPACK [2]. The latter was performed under periodic boundary conditions on domains with length 3 (the period of U^*) for 800 grid-points, and length 6 for 1600 grid-points. Since curves of essential spectrum connect eigenvalues from lengths 3 and 6, computing these spectra helps to locate isola of the essential spectrum.

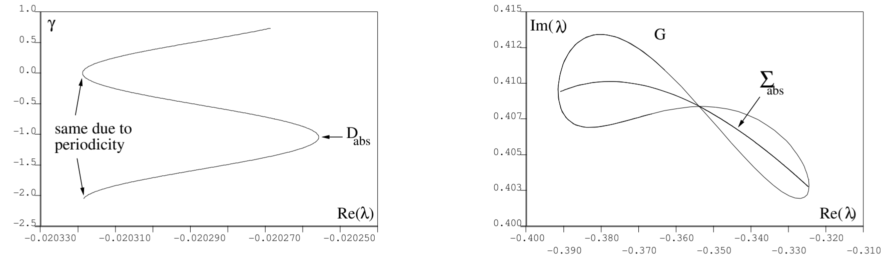
Here the isolas D, E, F, G lie in the boundary of the region Ω_∞, satisfy the assumptions of Theorem 4.5 and therefore contain absolute spectrum as plotted in Figures 5.3, 5.4 and 5.5, respectively. The isolas within the curves C and H do not have constant Morse index on their boundaries. While Theorem 4.5 cannot be applied directly, Lemma 4.2 implies that the self intersection points in C and H lie in curves of $\Sigma^1_{\rm abs}$, and these cross from Ω_∞ into the interior isolas, see also Figure 4.1. Since isola E and F are contained in the right half plane, the wave train is absolutely unstable.

Details of the essential spectrum and curves of Floquet exponents for the isola C and D are plotted in Figure 5.6. Each crossing point of two branches $\Re(\nu(\lambda))$ in Figure 5.6(b) lies in the generalized absolute spectrum. For $\Re(\lambda) > -0.05$ the real parts of the Floquet exponents are separated into two unstable and two stable exponents, as predicted for the region Ω_∞. Computations on the real line fail to indicate the instability of the small isola E and F in $\mathbb{C}\setminus\mathbb{R}$, see Figure 5.2.

As to generalized absolute spectrum, recall $\lambda \in \Sigma^j_{\rm abs}$, if $\Re(\nu_j(\lambda)) = \Re(\nu_{j+1}(\lambda))$ for $\Re(\nu_1) \geq \ldots \geq \Re(\nu_n)$. The spatial Morse index j is one plus the number of Floquet exponents with larger real part, e.g. in Figure 5.6(b) it is one plus the number of curves above a point in the generalized absolute spectrum.

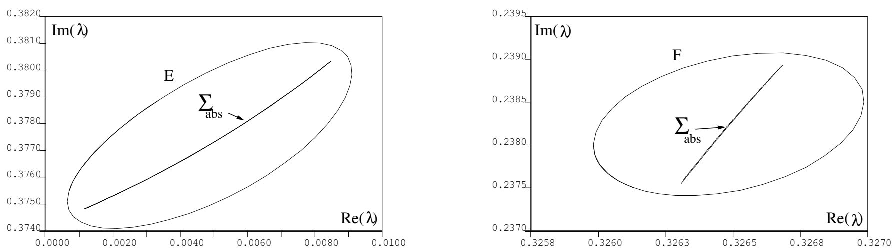
Due to symmetry, $\Sigma^*_{\rm abs} \cap \mathbb{R}$ consists of λ, where either two $\nu(\lambda)$ are complex conjugate, coincide (branch points), or a positive and negative Floquet *multiplier* have same modulus. In the latter case two Floquet exponents differ by the factor $e^{i\pi}$, which cannot occur for spatial eigenvalues of a uniform steady state. By symmetry, intervals in $\Sigma_{\rm abs} \cap \mathbb{R}$ with a complex conjugate pair terminate at branch points, while a curve of $\Sigma_{\rm abs}$ typically crosses $\mathbb{R}$ with vertical tangent at a point where multipliers in $\Sigma_{\rm abs} \cap \mathbb{R}$ have opposite sign, cf. [11] §4.3.

Figure 5.3 shows a curve of absolute spectrum (index $i_\infty = N = 2$) crossing the real line at the point D_{abs}, see also Figure 5.6(b). This curve was predicted above using Theorem 4.5, and was computed using the algorithm of its proof as described



(a) (b)

Fig. 5.4: (a) Parameterization of the curve of absolute spectrum from Figure 5.3 by $\gamma = \Im(\nu_2 - \nu_3)$ with period $2\pi/3 \sim 2.1$. For $\mathrm{D_{abs}}$ see also Figures 5.6 and 5.3. (b) Isola G and the imbedded curve of absolute spectrum.



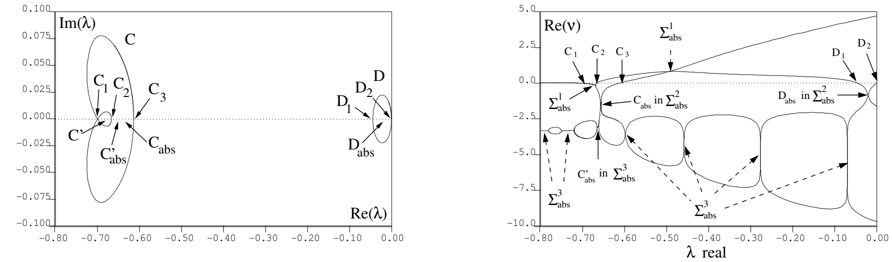
(a) (b)

Fig. 5.5: Isolas and imbedded curves of absolute spectrum. (a) isola E (b) isola F.

in Remark 3. Figure 5.4(a) shows the parameterization of this curve of absolute spectrum by the difference of imaginary parts $\kappa = \Im(\nu_2 - \nu_3)$. Since the period of U^* is 3, the period in κ is $2\pi/3 \sim 2.1$. There are branch points at $\kappa = j2\pi/3$, $j \in \mathbb{Z}$, the endpoints of the curve of absolute spectrum. At $\kappa = -\pi/3$, $j \in \mathbb{Z}$, the Floquet exponents are real, which corresponds to the point $\lambda = \mathrm{D_{abs}} \in \mathbb{R}$. Since the Floquet exponents' real parts plotted in Figure 5.6(b) differ for $\lambda \in \mathrm{int(D)} \cap \mathbb{R} \setminus \{\mathrm{D_{abs}}\}$, there is no further Σ^*_{abs} in D that intersects the real line.

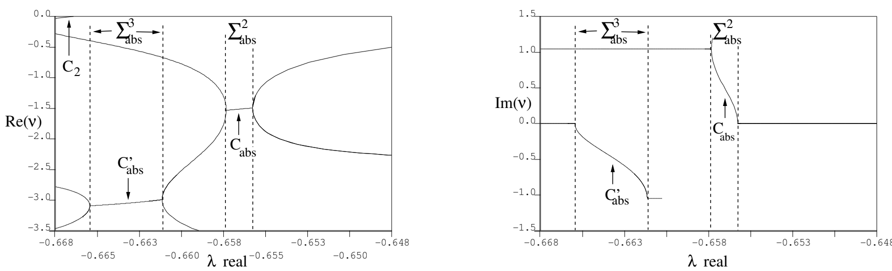
We next investigate some intervals of generalized absolute spectrum on the real axis. Since we strive for illustration, we disregard the points of Σ^*_{abs} that are marked with dotted arrows in Figure 5.6(b).

As mentioned above, Theorem 4.5 does not apply to isola C directly. However, for suitable exponential weights, see Remark 1, e.g. $\eta = 1.1$, part of isola C continues in η to a Jordan curve C"$\in$ C $\cup\Omega^\eta_\infty$. In this weighted space, Theorem 4.5 applies and predicts a curve of $\Sigma_{\mathrm{abs}} = \Sigma^2_{\mathrm{abs}}$ inside C", hence inside C. Indeed, continuing the branches of $\nu(\lambda)$ for $\lambda \in \mathbb{R}$ there is an interval of Σ_{abs} on the real line confined within isola C, see Figures 5.6 and 5.7. In addition, isola C contains an interval of Σ^3_{abs}. Figures 5.7(a) and (b) show that two real distinct Floquet exponents meet in branch points and become a complex conjugate pair at the endpoints of these intervals. Concerning other Morse indices, Figure 5.6(b) shows a branch point in $\Sigma^1_{\mathrm{abs}} \cap \mathbb{R}$ between the points $\mathrm{C_1}$ and $\mathrm{C_2}$ in isola C'. The attached interval of Σ^1_{abs}



(a) (b)

Fig. 5.6: (a) Isola C and D of essential spectrum, see also Figure 5.2. (b) Branches of Floquet exponents for $\lambda \in \mathbb{R}$ in the range of (a). Intersections with essential spectrum and generalized absolute spectrum can be read off as indicated, compare also the location of points with (a).



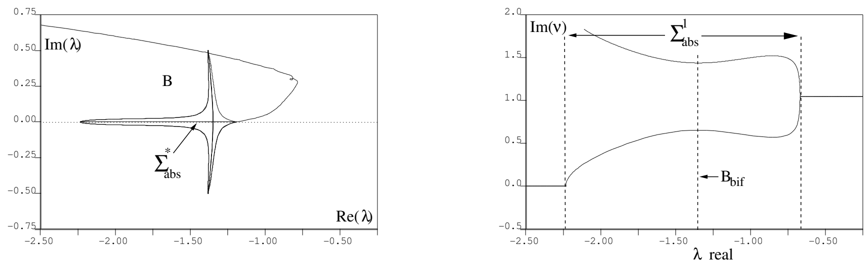
(a) (b)

Fig. 5.7: Branches of two spatial Floquet exponents within isola C for $\lambda \in \mathbb{R}$. Only one branch of a complex conjugate pair is plotted. (a) Real part, magnified from Figure 5.6(b). (b) Imaginary part with period $2\pi/3 \sim 2.1$; horizontal lines correspond to real spatial Floquet exponents.

crosses the self intersection point of isola C as predicted by Lemma 4.2 and extends into isola B, see Figure 5.8(a). The curves of imaginary parts in Figure 5.8(b) illustrate again how two Floquet exponents become a complex conjugate pair at the endpoints of the interval.

Without including a Figure we remark that isola A contains an interval of $\Sigma^1_{\text{abs}} \cap \mathbb{R}$ as predicted by Theorem 4.5. In fact, more intervals and crossing curves of generalized absolute spectrum on the real line occur as the branches of Floquet exponents repeatedly cross at the dotted arrows in Figure 5.6(b).

Finally, we expect the bifurcation of a curve of Σ^j_{abs} into the complex plane at a critical point of the imaginary part $\Im(\nu(\lambda))$ in any interval of $\Sigma^j_{\text{abs}} \in \mathbb{R}$ bounded by branch points, see [11] §4.2.3. More precisely, we expect a curve of Σ^*_{abs} crosses vertically at a point where $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Im(\nu(\lambda)) = 0$. Indeed, we can numerically locate such bifurcation points and switch to the bifurcating branch. An example for this in isola B is plotted in Figure 5.8.



(a) (b)

Fig. 5.8: We plot two curves of Σ^1_{abs} that cross within isola B at B_{bif}. (a) Spectrum in λ-plane. (b) Location of the crossing point at a local extremum of $\Im(\nu)$ for $\lambda \in \mathbb{R}$.

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