Abstracts

Singularities and intrinsic front dynamics of FitzHugh-Nagumo type systems

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This talk concerned existence, stability and bifurcation of 'front' interface solutions to certain perturbations of the prominent Allen-Cahn model for phase separation on the line $x \in \mathbb{R}$. As in the FitzHugh-Nagumo equations, we couple the Allen-Cahn equation to linear equations. Specifically, a seemingly weak coupling is considered that on 'small' ξ and 'large' $x = \xi/\varepsilon$ spatial scales gives the systems

$$\begin{aligned} U_t &= U_{\xi\xi} + U - U^3 - \varepsilon g(V, W; \mu) \\ \tau V_t &= V_{\xi\xi} + \varepsilon^2 (U - V) \\ \theta W_t &= D^2 W_{\xi\xi} + \varepsilon^2 (U - W) \end{aligned} \qquad \begin{aligned} U_t &= \varepsilon^2 U_{xx} + U - U^3 - \varepsilon g(V, W; \mu) \\ \frac{\tau}{\varepsilon^2} V_t &= V_{xx} + (U - V) \\ \frac{\theta}{\varepsilon^2} W_t &= D^2 W_{xx} + U - W, \end{aligned}$$

which are equivalent for $\varepsilon > 0$. Here μ is a set of parameters.

The large spatial scale $x = \xi/\varepsilon$ highlights the spatial scale separation for slowly travelling waves when $0 < \varepsilon \ll 1$. The question we adress is the impact of the slowly varying components V, W on the stable stationary Allen-Cahn fronts connecting ± 1. The system falls into the category of second order semi-strong interaction models [6] and indeed it turns out that front motion with velocity of order ε^2 arises; much faster than the metastable Allen-Cahn front interaction of exponentially small order [1, 2]. We illustrate that this occurs already due to the interaction of a *single front* with the spatially slowly varying 'fields' V, W.

For the linearly coupled, and thus *minimally nonlinear* case

$$\mu = (\alpha, \beta, \gamma), \; g(V, W; \mu) = \alpha V + \beta W + \gamma \tag{1}$$

the above system is a special parameter regime of a phenomenological gas-discharge model [7] and the existence, stability as well as interaction of multi-fronts has been studied already in [3, 4, 5]. However, the detailed analysis especially of the present regime of τ, θ becomes quite involved for multi-fronts. Here we present a complete picture of the stability of single fronts and the organizing center for existence: a butterfly catastrophe, which requires coupling to both V and W. Moreover, we prove a singularity imbedding when allowing general nonlinear coupling $g(V; \mu)$ already with only one additional component V (or W).

The existence of fronts near the Allen-Cahn fronts is a singular perturbation problem, which may be summarized as follows.

Theorem 0.1. *For any bounded set of μ, τ, θ, D, c there is ε_0 and an open neighborhood $\mathcal{U} \subset \mathbb{R}^6$ of the singular heteroclinic solution for $\varepsilon = 0$ connecting -1 to $+1$ (or vice versa) such that for all $0 < \varepsilon < \varepsilon_0$ solutions to*

$$\Gamma := g\left(\frac{c\tau}{\sqrt{c^2\tau^2+4}}, \frac{c\theta}{\sqrt{c^2\theta^2+4D^2}}; \mu\right) = 0 \tag{2}$$

are in one-to-one correspondence to front solutions with velocity $\varepsilon^2 c$ that lie in $\mathcal{U}$ and connect the perturbed homogeneous states $U = V = W = \pm 1$. These solutions form a smooth family, which converges uniformly to the singular heteroclinic solution as $\varepsilon \searrow 0$.

In addition, for $\gamma = c = 0$ there is one smooth subfamily of odd functions of ξ.

Based on this result we prove that in the minimally nonlinear case (1), the front existence problem is organized by a butterfly catastrophe, that is $\Gamma = \mathcal{O}(c^5)$. Specifically, this occurs if and only if

$$\gamma = 0,\ 2\sqrt{2}/3 = \alpha\tau + \beta\theta/D,\ \alpha D^3\tau^3 + \beta\theta = 0. \tag{3}$$

In particular, $\alpha\beta < 0$ is required so that both V and W are involved; otherwise the organizing center is a cusp. However, the unfolding is incomplete since Γ is an odd function of c and adding further linearly coupled linear equations do not complete the unfolding. A nonlinear symmetry breaking term such as ηV^2 in g is required to provide a versal unfolding. More generally, we prove that any singularity can be imbedded into the existence problem by suitable choice of g already when W is absent.

In addition to this existence analysis, we study stability of the fronts for the minimally nonlinear case, and derive via an Evans-function approach the following result.

Theorem 0.2. *Consider the case (1) and choose a parameter curve μ_ε so that a front exists for $0 < \varepsilon < \varepsilon_0$ as in Theorem 0.1. Then the critical eigenvalues of the linearization in the front are of the form $\lambda = \varepsilon^2\hat{\lambda} + \mathcal{O}(\varepsilon^{5/2})$, where $\hat{\lambda}$ is a root of*

$$E(\hat{\lambda}) := -\frac{\sqrt{2}}{6}\hat{\lambda} + \alpha\left(\frac{1}{\sqrt{c^2\tau^2+4}} - \frac{1}{\sqrt{c^2\tau^2+4(\hat{\lambda}\tau+1)}}\right) + \beta\left(\frac{1}{\sqrt{c^2\theta^2+4D^2}} - \frac{1}{\sqrt{c^2\theta^2+4D^2(\hat{\lambda}\theta+1)}}\right). \tag{4}$$

Moreover, E possesses at most two complex roots in addition to $\hat{\lambda} = 0$. At a butterfly catastrophe (3), E has a double root at $\hat{\lambda} = 0$ and the third (real) root has negative real part if and only if $\beta D^3 > -\alpha$.

Notably, the (slightly rewritten) existence problem $\Gamma = 0$ depends on θ/D, while the (slightly rewritten) E depends on β/D. However, the joint existence and stability problem depends on β, θ and D individually.

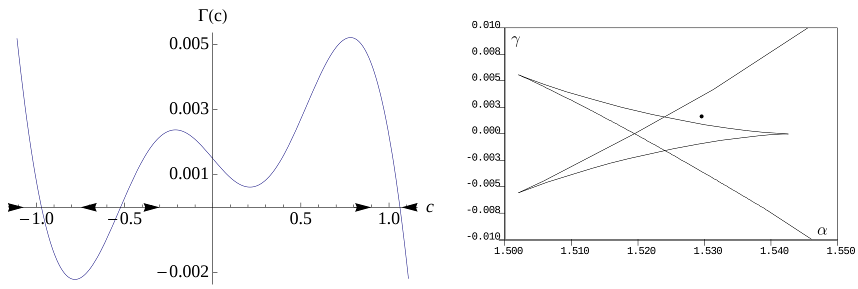

FIGURE 1. $\tau = 1$, $\theta = 2$, $\beta = -0.3$. (a) Graph of $\Gamma(c)$ for $\gamma = 0.0015$, $\alpha = 1.53$ with flow on center manifold illustrated by arrows; (b) curves of folds, bullet marks the location of (a).

Based on Theorem 0.2, for $\beta D^3 > -\alpha$, *we further prove a center manifold reduction to a scalar ODE* whose steady states correspond to fronts connected by heteroclinic orbits. The stable equilibria correspond to stable fronts in the PDE and *the heteroclinic connections describe accelerating, decelerating or even direction reversing fronts*. Due to Theorem 0.1 the organization of equilibria can be directly read off the graph of Γ as a function of c. More importantly, *the nature of velocity changes can be read off the graph of* Γ *directly*: the vector field is topologically given by $\dot{z} = \Gamma(z)$. See Figure 1 for an illustration.

More generally, the two nonzero eigenvalues of a front can also cross the imaginary axis and we thus expect a Hopf bifurcation, which is numerically corroborated. However, our computation of the normal form coefficients of the center manifold reduction is so far incomplete. In fact, the unfolding of the possible triple root of the Evans function is expected to contain a Bogdanov-Takens bifurcation with symmetry, which would provide a rich set of solutions.

The manuscripts with full details are in preparation for publication.

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