SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 13, No. 1, pp. 249–275

Quasi-Linear Parabolic Reaction-Diffusion Systems: A User's Guide to Well-Posedness, Spectra, and Stability of Travelling Waves*

M. Meyries†, J. D. M. Rademacher‡, and E. Siero§

Abstract. This paper is concerned with quasi-linear parabolic reaction-diffusion-advection systems on extended domains. Frameworks for well-posedness in Hilbert spaces and spaces of continuous functions are presented, based on known results using maximal regularity. It is shown that spectra of travelling waves on the line are meaningfully given by familiar tools for semilinear equations, such as dispersion relations, and basic connections of spectra to stability and instability are considered. In particular, a principle of linearized orbital instability for manifolds of equilibria is proven. Our goal is to provide easy access for practitioners to these rigorous methods. As a guiding example the Gray–Scott–Klausmeier model for vegetation-water interaction is considered in detail.

Key words. quasi-linear problem, reaction-diffusion, travelling waves, stability, orbital instability, maximal L^p regularity, Gray–Scott–Klausmeier model

AMS subject classifications. Primary, 35K57; Secondary, 35K59, 35C07, 35B35, 35B36

DOI. 10.1137/130925633

1. Introduction.

In this paper we present rigorous frameworks for well-posedness, spectra, and nonlinear stability of travelling wave solutions (pulses, fronts, and wavetrains) of quasi-linear parabolic reaction-diffusion systems of the form

$$u_t = (a(u)u_x)_x + f(u, u_x), \qquad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

with unknown $u(t, x) \in \mathbb{R}^N$. The nonlinearities a, f are smooth, and $a(u) \in \mathbb{R}^{N\times N}$ is strongly elliptic in the domain of interest but does not have to be symmetric. We further consider a variant of (1.1) in higher space dimensions $x \in \mathbb{R}^n$ up to $n = 3$. The nonlinearities may also depend explicitly on x in an appropriate way.

Quasi-linear reaction-diffusion systems arise as models in various contexts due to nonlinear fluxes, density-dependent diffusion, and self- or cross diffusion; see, e.g., [2,54,55]. For pattern formation problems it is natural to consider an extended domain and to neglect the influence of boundary conditions. Travelling waves, i.e., solutions of (1.1) constant in a co-moving frame $\xi = x - ct$ with speed $c \in \mathbb{R}$ having constant or periodic asymptotic states, are among

*Received by the editors June 16, 2013; accepted for publication (in revised form) by K. Promislow October 22, 2013; published electronically February 25, 2014. The second and third authors acknowledge support by the Complexity program of the Dutch research fund (NWO).
http://www.siam.org/journals/siads/13-1/92563.html
†Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik, 06099 Halle, Germany (martin.meyries@mathematik.uni-halle.de).
‡Universität Bremen, Fachbereich Mathematik, Postfach 33 04 40, 28359 Bremen, Germany (rademach@math.uni-bremen.de). This author's work was partially supported by the NWO cluster NDNS+ and by his previous employer, Centrum Wiskunde & Informatica (CWI), Amsterdam.
§Mathematisch Instituut, Universiteit Leiden, P.O. Box 9512, 2300 RA Leiden, The Netherlands (esiero@math.leidenuniv.nl).

the simplest interesting reaction-diffusion patterns and are observed for different types of quasi-linear systems; see, e.g., [27, 34, 41, 42, 45, 62, 70].

For semilinear parabolic problems on the line it is well known that, e.g., H^1 or BUC^1 are suitable phase spaces for well-posedness in a perturbative setting [13, 26]. The corresponding spectrum of the linearization is characterized in terms of the dispersion relation and the Evans function [23, 59]. In some situations, in particular when the essential spectrum does not touch the imaginary axis, nonlinear (orbital) stability of a wave can directly be deduced by a principle of linearized stability [26, 60]. An excellent reference for the spectrum and stability of nonlinear waves in the semilinear context is [28].

For quasi-linear models an analogous unified framework for well-posedness, spectra, and stability of waves seems less known. It seems that the majority of concrete well-posedness results in the literature concern bounded domains. Moreover, when the general results are formulated abstractly or under abstract conditions, a user needs to search for suitable function spaces and verify hypotheses that discourage rigor (even though some examples provide guidelines).

However, the spectrum of the linearization in a travelling wave can only be meaningfully determined based on a well-posedness setting. For instance, a Turing-instability determined via the usual dispersion relation lacks a basis without a consistent phase space. Conveniently, the pattern forming nature of a Turing-instability can be identified ad hoc since the existence of travelling wave patterns is an ODE problem. Well-posedness is, however, required to prove that a spectrally unstable solution is indeed unstable under the nonlinear evolution. Such a result then justifies the computation of stability boundaries by the spectrum, as in [53, 65] (see also section 5).

The purpose of this paper is to present rigorous settings for quasi-linear parabolic problems in the travelling wave context as described above. We aim for a presentation accessible to practitioners, in the spirit of [13, 26, 59] for semilinear problems. To this end we bring together and apply to (1.1) mostly abstract results from the different fields involved in well-posedness, spectra, and stability. This puts the naively expected analogy to the semilinear case on firm grounds. For quasi-linear systems, new difficulties mainly arise on a technical level concerning well-posedness and nonlinear stability. Most importantly, a variation-of-constants formula is not available. Further, when dealing with quasi-linear problems one has to take into account all available regularity as prescribed by sharp trace results such that in general one cannot take fractional power domains as a phase space for the solution semiflow. Instead one has to work with real interpolation spaces (see section 2.1) or the domain of the linearized operator itself. However, in the end it turns out that the familiar spaces H^2 and BUC^2 are possible phase spaces and that the spectral theory and the sufficient conditions for nonlinear stability are analogous to the semilinear case, at least in noncritical cases.

There are several abstract settings for well-posedness of general quasi-linear parabolic problems available in the literature (see [2, 5, 15, 24, 29, 33, 40, 49, 69], and [4] as well as section 2.3 for a selective overview). These have advantages and disadvantages depending on the present context, and the geometric (qualitative) theory is more or less developed in each case. On the other hand, solutions may be constructed by fixed point arguments tailor-made for the issues under investigation (e.g., [71]). The (real) viscous conservation laws are an important and well-studied class of quasi-linear problems, where well-posedness results exploit the additional structure [31]. We refer to the survey [72] and the references therein.

Our focus lies on the approach of [14, 33, 49] based on maximal L^p-regularity, but we also highlight the approach of [40] based on maximal Hölder regularity. Besides reaction-diffusion problems, the approach of [14,33,49] and its extensions apply successfully to the local theory of free boundary problems and to general parabolic problems with nonlinear boundary conditions. Here the geometric theory is well developed and still advances, especially for the needs in the context of free boundary problems. The approach of [40] also applies to fully nonlinear problems.

Recently, in [50, 51] the principle of linearized orbital stability with asymptotic phase for manifolds of equilibria has been established in the quasi-linear case, for any sufficiently strong well-posedness setting (see, e.g., [26, section 5.1] for the semilinear case). It in particular applies to the orbital stability of pulses and fronts for (1.1) in both approaches mentioned before. The conclusion from arbitrary unstable spectrum to nonlinear orbital instability of a manifold of equilibria does not seem to exist in the literature. Refining arguments from [26, Theorem 5.1.5] and [61] for single equilibria, we close this gap in the present paper. This might be of interest also in other contexts, where families of equilibria occur.

In more detail, our considerations may be summarized as follows:

- In one space dimension, $x \in \mathbb{R}$, a possible phase space for the evolution under (1.1) of localized perturbations from travelling wave and other pattern-type solutions is the Sobolev space H^2 (Theorem 2.4). For nonlocalized perturbations BUC^2 (C^2-functions, bounded and uniformly continuous with all derivatives) can be chosen as a phase space (Theorem 2.7).
- For arbitrary space dimensions $x \in \mathbb{R}^n$ the higher order Sobolev spaces H^k with $k > 1 + \frac{n}{2}$ are still possible phase spaces (see section 2.1.2). For $n \leq 3$ one can take certain Besov spaces, (real) interpolating between L^2 and H^2 (Theorem 2.5). In the latter case the advantage is that the linearization can be directly considered on L^2.
- The "spatial dynamics" spectral theory developed for semilinear parabolic systems on the line applies also in the quasi-linear case, which allows us to compute the spectrum of travelling waves in a familiar way (see section 3.3). In particular, the spectrum is independent of the chosen setting (Proposition 3.2).
- The well-known nonlinear stability result with asymptotic phase for travelling waves with simple zero eigenvalue applies in these settings (Proposition 4.1, as a direct consequence of [50, 51]).
- Without assuming a spectral gap or an unstable eigenvalue, it is shown that an unstable spectrum implies orbital instability of pulses and fronts (Theorem 4.3) and instability of wavetrains (Proposition 4.2). Here we rely on a general result on orbital instability of manifolds of equilibria (Lemma 4.4).

We emphasize that the divergence form (1.1) is assumed only in view of applications. In a smooth setting, the equation $u_t = a(u)u_{xx} + f(u, u_x)$ can be cast into divergence form by a suitable redefinition of a and f.

We believe that also the more general results in [57] on spectra of modulated travelling waves carry over to the quasi-linear case, but we do not enter into details here. Also the nonlinear stability of wavetrains is not considered. This is a delicate issue since zero always lies in the essential spectrum. Hence, the best one can hope for is heat-equation-like decay.

Under certain assumptions this has been established for the semilinear reaction-diffusion case in [19,60]. A special quasi-linear case, more precisely the quasi-linear integral boundary layer model, is considered in [25]. Also for viscous shocks the spectrum touches the origin, and stability in weighted spaces can be established. We refer to [73], the survey [72] and the references therein, as well as to [9] for more recent results.

In section 5 we illustrate our general considerations by means of the Gray–Scott–Klausmeier vegetation-water interaction model [32] for $x \in \mathbb{R}$ given by

$$\begin{aligned} w_t &= (w^2)_{xx} + Cw_x + A(1-w) - wv^2, \\ v_t &= Dv_{xx} - Bv + wv^2, \end{aligned} \tag{1.2}$$

with constants $A, B \geq 0$, $C \in \mathbb{R}$, and $D > 0$. This system is the original motivation for the present study. It is quasi-linear due to the porous medium term $(w^2)_{xx} = 2(ww_{xx} + (w_x)^2)$ and is therefore parabolic only in the regime $w > 0$, in which (1.2) supports a large family of travelling waves (see [65] and section 5).

This paper is organized as follows. In section 2 different well-posedness setting results for (1.1) are treated; section 3 is devoted to the spectrum of the linearization in travelling waves. The connection to nonlinear stability and instability is considered in section 4. In section 5 we expand the discussion of (1.2) and illustrate the application of the general results. For the sake of self-containedness we prove some technical results in the appendix.

Notation. All Banach spaces are real, and we consider complexifications if necessary. We write $\mathscr{L}(X_1, X_0)$ for the bounded linear operators between Banach spaces X_0, X_1, and $\mathscr{L}(X_0) = \mathscr{L}(X_0, X_0)$. The usual Sobolev spaces based on $L^p(\mathbb{R}^n)$ are denoted by $H^{k,p}$ and $H^k = H^{k,2}$. By $\mathrm{BC}^k = \mathrm{BC}^k(\mathbb{R}^n)$ and $\mathrm{BUC}^k = \mathrm{BUC}^k(\mathbb{R}^n)$ we denote the Banach space of bounded C^k-functions and of bounded C^k-functions such that all derivatives up to order k are uniformly continuous, respectively.

2. Frameworks for well-posedness.

We formulate the abstract well-posedness results based on maximal regularity and present three concrete frameworks for quasi-linear reaction-diffusion systems. In one space dimension we obtain well-posedness in H^2 and in BUC^2, and in space dimensions less than or equal to three we have well-posedness in certain Besov spaces. More general problems and further settings are briefly discussed at the end of this section.

2.1. Well-posedness based on maximal L^p-regularity.

We formulate the results of [33,49] for abstract quasi-linear parabolic problems of the form

$$\partial_t u = A(u)u + F(u), \quad t > 0, \qquad u(0) = u_0, \tag{2.1}$$

in a Hilbert space setting. Let X_0, X_1 be Hilbert spaces with X_1 continuously and densely embedded into X_0. Roughly speaking, X_0 is the base space for (2.1), and $A(u(t))$ is an unbounded linear operator on X_0 with (time-independent) domain X_1. It turns out that on this abstract level the phase space where the solution semiflow for (2.1) acts is a real interpolation space,

$$\mathcal{X} = (X_0, X_1)_{1-1/p,p}, \qquad p \in (1, \infty),$$

between X_0 and X_1. For a definition and the properties of these spaces we refer to the textbooks [11, 39, 66]. At this point we only note that $X_1 \subset \mathcal{X} \subset X_0$ and that $\mathcal{X}$ is in general *not* a Hilbert space, with exceptions for $p = 2$. Fortunately, explicit characterizations of $\mathcal{X}$ are possible in the concrete settings that we shall use later, e.g., $H^1 = (L^2, H^2)_{1/2,2}$. The real interpolation spaces are the analogues to the fractional power domains in semilinear theory [13, 26]. These two types of intermediate spaces between X_0 and X_1 differ, in general (again with exceptions for $p = 2$), but are closely related (see, e.g., [39, Proposition 4.1.7]).

Recall from [21,40] that a densely defined operator B on X_0 generates a strongly continuous analytic semigroup if and only if $\|\lambda(\lambda - B)^{-1}\|_{\mathscr{L}(X_0)}$ is uniformly bounded for λ from a left open sector in $\mathbb{C}$.

As a consequence of the results in [33, 49] we have the following.

Theorem 2.1. *Let $p \in (1, \infty)$ and $X_1 \subset \mathcal{X} \subset X_0$ be as above. Assume that there is an open set $\mathcal{V} \subseteq \mathcal{X}$ such that*

- *$F : \mathcal{V} \to X_0$ and $A : \mathcal{V} \to \mathscr{L}(X_1, X_0)$ are Lipschitz on bounded sets;*
- *for each $w_0 \in \mathcal{V}$, the operator $A(w_0)$ with domain X_1 generates a strongly continuous analytic semigroup on X_0.*

Then (2.1) is locally well-posed in $\mathcal{V}$, with solutions in a strong L^p sense.

More precisely, the theorem yields solvability of (2.1) as follows. For each initial value $u_0 \in \mathcal{V}$ there is a maximal existence time $t^+(u_0) > 0$ and a unique solution $u = u(\cdot; u_0) \in C([0, t^+(u_0)), \mathcal{V})$ of (2.1) such that $u \in H^{1,p}(J, X_0) \cap L^p(J, X_1)$ for time intervals $J = (0, T)$ with $T < t^+(u_0)$. Here $H^{1,p}(J, X_0)$ denotes a vector-valued Sobolev space, which is defined as in the scalar-valued case. An equivalent norm on $H^{1,p}(J, X_0) \cap L^p(J, X_1)$ is given by

$$\|u\|^p_{H^{1,p}(J,X_0)\cap L^p(J,X_1)} = \int_J \left(\|u'(t)\|^p_{X_0} + \|u(t)\|^p_{X_1}\right) dt.$$

Furthermore, $t^+(u_0)$ is finite only if either $\operatorname{dist}(u(t; u_0), \partial\mathcal{V}) \to 0$ or $\|u(t; u_0)\|_{\mathcal{X}} \to \infty$ as $t \to t^+(u_0)$. The map $t^+ : \mathcal{V} \to (0, \infty]$ is lower semicontinuous, and the local solution semiflow, $(t, u_0) \mapsto u(t; u_0)$, is continuous with values in $\mathcal{V} \subseteq \mathcal{X}$. If F and A are smooth, then the semiflow enjoys smoothness properties as well. We demonstrate this in Proposition A.4 in the appendix for a neighborhood of a steady state.

Note that if $A(w_0)$ generates an analytic semigroup for $w_0 \in \mathcal{X}$, then the Lipschitz property of A as in the theorem combined with well-known perturbation results for semigroups (see [40, Proposition 2.4.2]) imply that this is true for any $A(\tilde{w}_0)$ with $\tilde{w}_0$ in a small neighborhood of w_0. This gives a candidate for $\mathcal{V}$.

To verify the assumptions in [33, section 2] and [49, Theorem 3.1] and prove Theorem 2.1, we need only to know that $-A(w_0)$ has for each $w_0 \in \mathcal{V}$ the property of maximal L^p-regularity on finite time intervals J. But in *Hilbert spaces* this already follows from the assumed generator property of $A(w_0)$. Indeed, by [20, Theorems 3.3, 7.1] it suffices to consider the case $p = 2$, $J = \mathbb{R}_+$, and that the semigroup generated by $A(w_0)$ is exponentially decaying. In this situation maximal L^2-regularity follows from [63] (see also [49, Theorem 1.6] for the short proof using Plancherel's theorem).

2.1.1. One space dimension: Well-posedness in H^2. For $u(t, x) \in \mathbb{R}^N$ we apply the abstract result Theorem 2.1 to the reaction-diffusion system

$$u_t = (a(u)u_x)_x + f(u, u_x), \qquad t > 0,\ x \in \mathbb{R}. \tag{2.2}$$

To obtain a simple setting with familiar function spaces which is at the same time directly linked to L^2-spectral theory, we work with $X_0 = H^1 = H^1(\mathbb{R})^N$ as a base space. In one space dimension (and only there) this is possible since H^1 is an algebra, i.e., $uw \in H^1$ and $\|uv\|_{H^1} \leq C\|u\|_{H^1}\|v\|_{H^1}$ for $u, w \in H^1$.

We start with the case when the nonlinearities in (2.2) are everywhere defined. We emphasize that a does not have to be symmetric, and that a, f may be less regular than actually stated.

Theorem 2.2. *Assume that $a : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is C^4 such that $a(\zeta) \in \mathbb{R}^{N\times N}$ is positive definite for each $\zeta \in \mathbb{R}^N$ and that $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is C^3 with $f(0, 0) = 0$.*

Then (2.2) is locally well-posed in the phase space $\mathcal{X} = H^2$. The solutions belong to $H^1(J, H^1(\mathbb{R})) \cap L^2(J, H^3(\mathbb{R})) \cap C(\overline{J}, H^2(\mathbb{R}))$ on time intervals $J = (0, T)$ away from the maximal existence time.

Proof. We choose $X_0 = H^1$, $X_1 = H^3$, and $p = 2$. Then $\mathcal{X} = (H^1, H^3)_{1/2,2} = H^2$; see [66, Remark 2.4.2/2]. Define the superposition (Nemytskii) operators A and F by $A(u)v = (a(u)v_x)_x$ and $F(u) = f(u, u_x)$. Then $F : H^2 \to H^1$ and $A : H^2 \to \mathscr{L}(H^3, H^1)$ are Lipschitz on bounded sets by Lemma A.1. For the generator property, let $w_0 \in H^2$ be arbitrary. Denote by A_{L^2} the realization of $A(w_0)$ on L^2, with domain H^2. Since $w_0, a(w_0) \in \mathrm{BC}^1$ by Sobolev's embedding $H^1 \subset \mathrm{BC}$, it follows from [6, Corollary 9.5] that the operator A_{L^2} generates an analytic C_0-semigroup on L^2. Next, let A_{H^1} be the realization of $A(w_0)$ on H^1, i.e., the restriction of A_{L^2} to H^1. Since $H^1 = (L^2, H^2)_{1/2,2}$ (see again [66]), it follows from [39, Theorem 5.2.1] that A_{H^1} with domain $D(A_{H^1}) = \{u \in H^2 : A_{L^2}u \in H^1\}$ generates an analytic C_0-semigroup as well. Using the algebra property of H^1, it is elementary to check that $D(A_{H^1}) = H^3$ (see the proof of Lemma A.3 in the appendix). Thus Theorem 2.1 applies. ■

Remark 2.3. Employing, e.g., *Angenent's parameter trick* (see [49, Theorem 5.1] and [22]), one can show that for smooth nonlinearities the solutions of (2.2) are smooth in space and time.

When investigating the stability of a nonlocalized travelling wave with respect to localized perturbations, one is led to a variant of (2.2) with x-dependent nonlinearities. Furthermore, in many situations the nonlinearities are not everywhere defined on $\mathbb{R}^N$, or the leading coefficient a is positive definite only in a subset of $\mathbb{R}^N$. For instance, this is the case for the Gray–Scott–Klausmeier model (1.2), where the focus lies on perturbations of travelling wave solutions in the parabolic regime $w > 0$.

For a general formulation, let $\overline{u} \in \mathrm{BC}^2(\mathbb{R}, \mathbb{R}^N)$ be a steady state of (2.2), i.e.,

$$(a(\overline{u})\overline{u}_x)_x + f(\overline{u}, \overline{u}_x) = 0. \tag{2.3}$$

Then $\overline{u} + u$ solves (2.2) for a perturbation u if and only if u solves

$$u_t = (a(\overline{u} + u)u_x)_x + (a(\overline{u} + u)\overline{u}_x)_x + f(\overline{u} + u, \overline{u}_x + u_x). \tag{2.4}$$

For this perturbative setting we have the following variant of Theorem 2.2. Here and in the following, the image of $\overline{u}$ is meant to be the set $\{\overline{u}(x) : x \in \mathbb{R}\}$.

Theorem 2.4. *Let $\overline{u} \in \mathrm{BC}^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (2.3), and let $U_1, U_2 \subseteq \mathbb{R}^N$ be open neighborhoods of the closure of the images of $\overline{u}$ (resp., $\overline{u}_x$). Assume that $a : U_1 \to \mathbb{R}^{N\times N}$ is C^4 such that $a(\zeta)$ is positive definite for any $\zeta \in U_1$, and $f : U_1 \times U_2 \to \mathbb{R}^N$ is C^3.*

Then there is an open neighborhood $\mathcal{V}$ of the zero function in H^2 such that (2.4) is locally well-posed in $\mathcal{V}$. If $U_1 = U_2 = \mathbb{R}^N$, then one can take $\mathcal{V} = H^2$.

Proof. Let again $X_0 = H^1$, $X_1 = H^3$, and $p = 2$ such that $\mathcal{X} = H^2$. Define

$$A(u)v = (a(\overline{u}+u)v_x)_x, \qquad F(u) = (a(\overline{u}+u)\overline{u}_x)_x + f(\overline{u}+u, \overline{u}_x + u_x). \tag{2.5}$$

Using $F(0) = 0$, Lemma A.1 yields $\mathcal{V} \subseteq H^2$ such that $F : \mathcal{V} \to H^1$ and $A : \mathcal{V} \to \mathscr{L}(H^3, H^1)$ are Lipschitz on bounded sets. If $\mathcal{V}$ is sufficiently small, then for each $w_0 \in \mathcal{V}$ the leading coefficient $a(\overline{u} + w_0)$ of $A(w_0)$ is positive definite, uniformly in $x \in \mathbb{R}$. Thus, as in the proof of Theorem 2.2, it follows from [6, Corollary 9.5] and an interpolation argument that $A(w_0)$ with domain H^3 has the required generator property on H^1 to apply Theorem 2.1. ■

2.1.2. Well-posedness in space dimensions $n \leq 3$. For simplicity, on $\mathbb{R}^n$ we consider quasi-linear reaction-diffusion-advection problems (using the sum convention):

$$u_t = \partial_i(a_{ij}(u)\partial_j u) + c_i \partial_i u + f(u), \qquad x \in \mathbb{R}^n. \tag{2.6}$$

Here, essentially, $a_{ij} : \mathbb{R}^N \to \mathbb{R}^{N\times N}$, $c_i \in \mathbb{R}^{N\times N}$ for $i, j = 1, \ldots, n$ and $f : \mathbb{R}^N \to \mathbb{R}^N$.

The approach of the previous subsection works in any dimension if one takes $X_0 = H^k(\mathbb{R}^n, \mathbb{R}^N)$ with $k > \frac{n}{2}$ as a base space, since then H^k is an algebra and the superposition operators are Lipschitz as before. This leads to the Hilbert space $\mathcal{X} = H^{k+1}$ as a phase space.

We present another functional analytic setting with $X_0 = L^2$ as a base space, for which Theorem 2.1 applies to (2.6) in space dimensions $n \leq 3$. The price one has to pay in the maximal L^p-regularity approach is that the phase space $\mathcal{X} = (L^2, H^2)_{1-1/p,p}$ becomes slightly more complicated to describe. It is the N-fold product $B^s_{2,p}$ of a *Besov space* $B^s_{2,p}(\mathbb{R}^n)$, with $s > 0$ and $p \in (1, \infty)$. For $s \notin \mathbb{N}$, it follows from [67, Theorem 2.6.1] that an equivalent norm for this space is given by

$$\|u\|_{B^s_{2,p}} = \|u\|_{H^k} + \sum_{|\alpha|\leq k} \left(\int_{|h|\leq 1} |h|^{-(s-k)p-n} \|D^\alpha u(\cdot + h) - D^\alpha u(\cdot)\|^p_{L^2}\, dh \right)^{1/p},$$

where k is the largest integer smaller than s. The Besov spaces are closely related to the more common Bessel-potential spaces H^s. For any $\varepsilon > 0$ we have the dense inclusions $H^{s+\varepsilon} \subset B^s_{2,p} \subset H^{s-\varepsilon}$. However, $B^s_{2,p} = H^s$ if and only if $p = 2$, and furthermore $B^s_{2,p}$ is a Hilbert space only for $p = 2$. Essential for the applications are the Sobolev embeddings

$$B^s_{2,p}(\mathbb{R}^n) \subset \mathrm{BC}(\mathbb{R}^n) \quad \text{for } s > \frac{n}{2}, \quad B^s_{2,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \text{for } s \geq \frac{n}{2} - \frac{n}{q} > 0. \tag{2.7}$$

These are a consequence of $B^s_{2,p} \subset H^{s-\varepsilon}$ and the corresponding embeddings for the H-spaces. For these and many more properties of B-spaces we refer to [66].

As above we consider a perturbative setting. Analogous to (2.4), for perturbations u of a steady state $\overline{u} \in \mathrm{BC}^2(\mathbb{R}^n, \mathbb{R}^N)$ of (2.6), one is lead to

$$\partial_t u = \partial_i(a_{ij}(\overline{u}+u)\partial_j u) + \partial_i(a_{ij}(\overline{u}+u)\partial_j \overline{u}) + c_i\partial_i(u + \overline{u}) + f(\overline{u}+u). \tag{2.8}$$

Note that the following well-posedness result in particular applies to (2.6) when setting $\overline{u}=0$ and assuming $f(0)=0$. Again no symmetry properties of the diffusion coefficients (a_{ij}) are required.

Theorem 2.5. *Let $n=1,2,3$. Let $\overline{u}\in \mathrm{BC}^2(\mathbb{R}^n,\mathbb{R}^N)$ be a steady state of (2.6), and let $U\subseteq\mathbb{R}^N$ be an open neighborhood of the closure of its image. For all $i,j=1,\dots,n$ assume that $c_i\in\mathbb{R}^{N\times N}$ is constant, that $a_{ij}:U\to\mathbb{R}^{N\times N}$ and $f:U\to\mathbb{R}^N$ are C^2, and that $a_{ij}(\zeta)$ is positive definite for any $\zeta\in U$.*

Then for all sufficiently large $p\in(2,\infty)$ there is an open neighborhood $\mathcal{V}$ of the zero function in $B^{2-2/p}_{2,p}=B^{2-2/p}_{2,p}(\mathbb{R}^n)^N$ such that (2.8) is locally well-posed in $\mathcal{V}$. The solutions belong to $H^{1,p}(J,L^2)\cap L^p(J,H^2)\cap C(\overline{J},\mathcal{V})$ on time intervals J away from the maximal existence time. If $U=\mathbb{R}^N$, then one can take $\mathcal{V}=B^{2-2/p}_{2,p}$.

Proof. The choice $X_0=L^2$ and $X_1=H^2$ leads to $B^{2-2/p}_{2,p}=\mathcal{X}=(X_0,X_1)_{1-1/p,p}$ for $p\in(1,\infty)$; see [66, Remark 2.4.2/4]. Let $A(u)v=\partial_i(a_{ij}(\overline{u}+u)\partial_j v)$, and denote by $F(u)$ the remaining terms on the right-hand side of (2.8). The Lipschitz properties of A and F on a neighborhood $\mathcal{V}$ of zero follow from Lemma A.2. For $w_0\in\mathcal{V}$ the operator $A(w_0)$ is elliptic, the coefficients are bounded, and the leading coefficient is uniformly Hölder continuous, since $B^{2-2/p}_{2,p}$ even embeds into BC^σ for some $\sigma>0$ if $n\le 3$ and p is large; see [66, Theorem 2.8.1]. Now the generator property on L^2 follows again from [6, Corollary 9.5]. ∎

The proof shows that if (2.8) is semilinear, then one can apply the results of [13, 26] and take a fractional power domain of the Laplacian as a phase space $\mathcal{X}$. This results in a fractional order Bessel-potential space $\mathcal{X}=H^s$ with s sufficiently close to 2, which is still a Hilbert space.

2.2. Well-posedness based on maximal Hölder regularity.

We formulate the well-posedness result of [40, Chapter 8] for abstract quasi-linear parabolic problems

$$\partial_t u=A(u)u+F(u),\quad t>0,\qquad u(0)=u_0. \tag{2.9}$$

The approach of [40] is based on maximal Hölder regularity (see also [3, Chapter III.2] for the general linear theory). It also covers fully nonlinear problems and does not take into account the quasi-linear structure of (2.9). It has the big advantage of being applicable in arbitrary Banach spaces X_0, while in applications maximal L^p-regularity is usually restricted to reflexive Banach spaces, excluding spaces of continuous functions. Moreover, the phase space equals the domain of the linearized operator, which is usually easier to describe than an interpolation space.

The following well-posedness result for (2.9) is a consequence of [40, Theorem 8.1.1, Proposition 8.2.3, Corollary 8.3.3].

Theorem 2.6. *Let X_0,X_1 be arbitrary Banach spaces such that X_1 is continuously and densely embedded in X_0. Let $\mathcal{V}\subseteq\mathcal{X}:=X_1$ be open, define $\mathcal{F}(u)=A(u)u+F(u)$, and suppose that*

- *$\mathcal{F}\in C^1(\mathcal{V},X_0)$ with locally Lipschitz derivative;*
- *for each $w_0\in\mathcal{V}$, the operator $\mathcal{F}'(w_0)$ with domain X_1 generates a strongly continuous analytic semigroup on X_0, and $\|u\|_{X_0}+\|\mathcal{F}'(w_0)u\|_{X_0}$ defines an equivalent norm on X_1.*

Then (2.9) is locally well-posed in $\mathcal{V}$, and solutions are classical in time.

As already mentioned, the phase space $\mathcal{X}$ is now a subset of X_1 and not of an intermediate space between X_0 and X_1. Well-posedness is similar to that of Theorem 2.1. The maximal existence time is lower semicontinuous, and the solution semiflow is continuous with values in $\mathcal{V}$. For each $\alpha \in (0,1)$ and an initial value $u_0 \in \mathcal{V}$, one obtains a unique maximal solution u of (2.9) such that $u \in \mathrm{BUC}^{1+\alpha}_{\alpha}([0,T], X_0) \cap \mathrm{BUC}^{\alpha}_{\alpha}([0,T], X_1)$ for $T < t^+(u_0)$. Here $\mathrm{BUC}^{\alpha}_{\alpha}$ is a weighted Hölder space; see [3, Chapter III.2] and [51, Example 3]. (It is slightly confusing that these spaces differ from the ones in [40] denoted by C^{α}_{α}, but $\mathrm{BUC}^{\alpha}_{\alpha}$ is indeed the regularity obtained in [40, Theorem 8.1.1].)

Theorem 2.6 applies to (2.2), (2.4), and (2.8) under assumptions similar to those of Theorems 2.2, 2.4, and 2.5, with different phase spaces. In particular, instead of a Besov space one obtains H^2 as a phase space in the setting of Theorem 2.5. We do not formulate the precise results but rather consider a setting for reaction-diffusion systems which is not covered by the approach of Theorem 2.1.

2.2.1. One space dimension: Well-posedness in BUC^2. We reconsider the case of one space dimension, i.e., for $u(t,x) \in \mathbb{R}^N$ the problem

$$u_t = (a(u)u_x)_x + f(u, u_x), \qquad t > 0,\ x \in \mathbb{R}. \tag{2.10}$$

We present a setting in which nonlocalized perturbations of steady states can be treated. For $k \in \mathbb{N}_0$, denote by $\mathrm{BUC}^k = \mathrm{BUC}^k(\mathbb{R}, \mathbb{R}^N)$ the Banach space of bounded uniformly continuous functions, endowed with the usual C^k-norm. It is shown in [40] that a scalar second order elliptic operator on $\mathrm{BUC} = \mathrm{BUC}^0$ behaves well and generates an analytic semigroup. This is the main ingredient for applying Theorem 2.6 as follows. The triangular structure of a is assumed for simplicity.

Theorem 2.7. *Let $\overline{u} \in \mathrm{BUC}^2(\mathbb{R}, \mathbb{R}^N)$ be a steady state of (2.10), and let $U_1, U_2 \subseteq \mathbb{R}^N$ be open neighborhoods of the closure of image of $\overline{u}$ (resp., $\overline{u}_x$). Assume that $a : U_1 \to \mathbb{R}^{N\times N}$ and $f : U_1 \times U_2 \to \mathbb{R}^N$ are C^2 such that*

- *for each $\zeta \in U_1$ the matrix $a(\zeta)$ is triangular, and the diagonal entries of a are positive and bounded away from zero uniformly.*

Then there is an open neighborhood $\mathcal{V}$ of $\overline{u}$ in BUC^2 such that (2.10) is locally well-posed in $\mathcal{V}$. One can take $\mathcal{V} = \mathrm{BUC}^2$ if $U_1 = U_2 = \mathbb{R}^N$.

Proof. Choose an open set $V \subset \mathbb{R}^N$ that contains the image of $\overline{u}$ and satisfies $\overline{V} \subset U$. Define $\mathcal{V}$ as the set of all $w_0 \in \mathrm{BUC}^2$ with image contained in V. Then $\mathcal{F}(u) = (a(u)u_x)_x + f(u, u_x)$ defines a superposition operator $\mathcal{F} : \mathcal{V} \to \mathrm{BUC}$. It is straightforward to check that $\mathcal{F} \in C^1(\mathcal{V}, \mathrm{BUC})$. At $w_0 \in \mathcal{V}$ we have

$$\mathcal{F}'(w_0)v = (a(w_0)v_x)_x + (a'(w_0)[(w_0)_x, v])_x + cv_x + f'(w_0)v, \qquad v \in \mathrm{BUC}^2,$$

and $\mathcal{F}' : \mathcal{V} \to \mathscr{L}(\mathrm{BUC}^2, \mathrm{BUC})$ is locally Lipschitz. For the generator property, let $w_0 \in \mathcal{V}$ be given. By [40, Corollary 3.1.9], each of the scalar-valued operators $v \mapsto a_{ii}(w_0)v_{xx}$ with domain BUC^2 generates an analytic C_0-semigroup on BUC, where a_{ii} are for $i = 1, \ldots, N$ the diagonal entries of a. Using the matrix generator result [46, Corollary 3.3] and the triangular structure of a, we conclude that the principle part $v \mapsto a(w_0)v_{xx}$ of $\mathcal{F}'(w_0)$ is a generator on $\mathrm{BUC}(\mathbb{R}, \mathbb{R}^N)$, with domain $\mathrm{BUC}^2(\mathbb{R}, \mathbb{R}^N)$. The remaining lower order terms preserve this property. The equivalence of the graph norm of $\mathcal{F}'(w_0)$ and the C^2-norm follows from the boundedness of the coefficients and the open mapping theorem. Therefore Theorem 2.6 applies to (2.10). ■

2.3. More general problems and other frameworks. The above results also hold for smooth x-dependent nonlinearities, provided the principal term a is positive definite uniformly in x. Also nonautonomous and nonlocal problems can be treated; see [4, 5, 40, 49]. Only the mapping properties of the superposition operators and the generator properties of the linearization are relevant. Both frameworks cover general quasi-linear systems in any dimension if one works with $X_0 = L^q$ for large q as a base space, since then the superposition operators are well-defined by Sobolev embeddings. Theorem 2.6 also allows us to work in spaces of Hölder continuous functions, L^∞, or subspaces of BUC like C_0 or $C(\overline{\mathbb{R}})$, based on the analytic generator results of [40] and [21, section VI.4].

A framework with spatial weights might also be of interest, for instance, to force some decay of solutions [71] or to treat singular terms [42]. Here in particular weights with exponential growth are straightforward to treat, as the generator results can be obtained from the unweighted case by a simple similarity transformation. Concerning weights, we also mention that the approach of [38] has proven useful for quasi-linear parabolic problems in weighted spaces; see [8, 25, 68]. Invariant manifolds for quasi-linear parabolic systems with nonlinear boundary conditions on bounded or exterior domains are constructed, e.g., in [36, 37, 64]; see also the references given there.

Besides the above approaches based on maximal L^p- and Hölder regularity there is a similar abstract approach based on continuous regularity [7, 15]. Completely different frameworks for problems in weaker settings on bounded domains with boundary conditions are presented in [2, 24]. They should also be applicable to problems on $\mathbb{R}^n$. Finally, the pioneering work of [35] should be mentioned. For a comprehensive overview of possible settings for quasi-linear parabolic problems we refer the reader to [4].

3. Stability and spectra of travelling waves. While travelling waves also occur in higher space dimensions, we restrict here to $x \in \mathbb{R}$.

Throughout, let $u_*(t, x) = \overline{u}(x - ct)$ be a travelling wave solution of

$$u_t = (a(u)u_x)_x + f(u, u_x), \qquad x \in \mathbb{R},$$

with speed $c \in \mathbb{R}$ and profile $\overline{u} \in \mathrm{BC}^\infty(\mathbb{R}, \mathbb{R}^N)$ solving the ODE (2.3). We assume that a, f are C^∞ and that a is uniformly positively definite in a vicinity of the image of $\overline{u}$. Suitable finite regularity of $\overline{u}, a, f$ suffices for each of the following results, and we assume infinite smoothness only for the sake of a simple exposition. We further assume that $\overline{u}$ is constant or periodic at infinity and that the asymptotic states are approached exponentially. A travelling wave is called a *pulse* or a *front* if the asymptotic states are equal or different homogeneous equilibria, respectively. A *wavetrain* is a periodic travelling wave, and we refer to travelling waves with at least one periodic asymptotic state as *generalized* fronts or pulses.

3.1. Stability in a perturbative setting. The evolution of perturbations u of u_* is governed by

$$u_t = (a(\overline{u} + u)u_x)_x + (a(\overline{u} + u)\overline{u}_x)_x + c(\overline{u}_x + u_x) + f(\overline{u} + u, \overline{u}_x + u_x), \tag{3.1}$$

where the co-moving frame $x - ct$ is again denoted by x. By translation invariance of the underlying equation, stability must be considered with respect to the family of translates

$$S = \{\overline{u}(\cdot + \tau) - \overline{u} : \tau \in \mathbb{R}\}.$$

Theorems 2.4, 2.5, and 2.7 guarantee local well-posedness of (3.1) for initial data from $\mathcal{X} = H^2$, $\mathcal{X} = B^{2-2/p}_{2,p}$, or $\mathcal{X} = \mathrm{BUC}^2$ sufficiently close to S. (Note that in Theorem 2.5 it is actually assumed that f is independent of u_x.) Even though $H^2 \subset B^{2-2/p}_{2,p}$, we distinguish between these cases, because of the different corresponding base spaces H^1 and L^2, and to highlight that a pure Sobolev space setting suffices for (3.1). For $\mathcal{X} = \mathrm{BUC}^2$, or in case of a pulse, one could equivalently consider (3.1) with $\overline{u}$ replaced by zero, in a neighborhood of $\{\overline{u}(\cdot + \tau) : \tau \in \mathbb{R}\}$.

If u_* is a pulse or a front, then S is in each setting a family of equilibria of (3.1).

Definition 3.1. *A pulse or front solution u_* is called* orbitally stable *if for $\varepsilon > 0$ there is $\delta > 0$ such that for $u_0 \in \mathcal{X}$ with $\mathrm{dist}_{\mathcal{X}}(u_0, S) \le \delta$ the corresponding solution u of (3.1) exists globally in time and satisfies $\mathrm{dist}_{\mathcal{X}}(u(t), S) \le \varepsilon$ for all $t > 0$. u_* is called* orbitally stable with asymptotic phase *if it is orbitally stable and if for each $u_0 \in \mathcal{X}$ sufficiently close to S there is τ_∞ such that the corresponding solution of (3.1) converges to $\overline{u}(\cdot + \tau_\infty) - \overline{u}$ as $t \to \infty$. u_* is* orbitally unstable *if it is not orbitally stable.*

For a wavetrain, translates of the profile cannot be realized by localized perturbations. Thus only for $\mathcal{X} = \mathrm{BUC}^2$ can orbital stability as above be considered. For localized perturbations, i.e., $\mathcal{X} = H^2$ or $\mathcal{X} = B^{2-2/p}_{2,p}$, stability of a wavetrain is understood with respect to stability of the zero solution of (3.1).

3.2. The spectrum of the linearization. The linearization $\mathcal{L}$ of the right-hand side of (3.1) in $u = 0$ is

$$\mathcal{L}\varphi = \alpha\varphi_{xx} + \beta\varphi_x + \gamma\varphi, \tag{3.2}$$

with smooth coefficients $\alpha(x), \beta(x), \gamma(x) \in \mathbb{R}^{N\times N}$ given by

$$\alpha = a(\overline{u}), \qquad \beta = a'(\overline{u})[\overline{u}_x, \cdot] + a'(\overline{u})[\cdot, \overline{u}_x] + c + \partial_2 f(\overline{u}, \overline{u}_x),$$

$$\gamma = a''(\overline{u})[\overline{u}_x, \cdot, \overline{u}_x] + a'(\overline{u})[\cdot, \overline{u}_{xx}] + \partial_1 f(\overline{u}, \overline{u}_x).$$

Depending on the chosen well-posedness framework, the operator $\mathcal{L}$ is considered on $X_0 = H^1$, L^2, or BUC, with domain H^3, H^2, or BUC^2, where we write $\mathcal{L}_{X_0}$ for a realization. The *spectrum* of $\mathcal{L}_{X_0}$ is the set of $\lambda \in \mathbb{C}$, where $\mathcal{L}_{X_0} - \lambda$ is not boundedly invertible. It is denoted by $\mathrm{spec}\,\mathcal{L}_{X_0}$.

As in the approach surveyed in [28, 59], we distinguish between the *point spectrum*, i.e., $\lambda \in \mathrm{spec}\,\mathcal{L}_{X_0}$ such that $\mathcal{L}_{X_0} - \lambda$ is a Fredholm operator of index zero, and its complement within the spectrum, called the *essential spectrum*. We will see that point and essential spectrum are independent of the chosen framework and that the familiar spectral theory for ordinary differential operators based on exponential dichotomies, as described in [28, 59], applies to $\mathcal{L}$.

Usually, the *set of eigenvalues* of $\mathcal{L}_{X_0}$ is called the point spectrum. Note that, with the above definition, eigenvalues can be contained in the essential spectrum. Moreover, eigenvalues are not independent of the setting. For instance, the operator $\partial_x - \mathrm{i}$ has a zero eigenvalue with eigenfunction $\phi(x) = e^{\mathrm{i}x}$ on BUC, but it is injective on L^2 and H^1. Of course this does not contradict Proposition 3.2 on kernel dimensions below since the operator is not Fredholm.

Since it is assumed that a is positive definite in a neighborhood of the image of $\overline{u}$, the multiplication by α^{-1} is an isomorphism in each setting. Thus the invertibility and Fredholm properties of $\mathcal{L} - \lambda$ are the same as for

$$\tilde{\mathcal{L}}(\lambda) = \alpha^{-1}(\mathcal{L} - \lambda) = \partial_{xx} + \alpha^{-1}\beta\partial_x + \alpha^{-1}(\gamma - \lambda),$$

which has constant leading order coefficients. As before, we write $\tilde{\mathcal{L}}_{X_0}(\lambda)$ for a realization of $\tilde{\mathcal{L}}(\lambda)$. The key to the spectral properties of $\tilde{\mathcal{L}}(\lambda)$ is the corresponding first order operator

$$\tilde{\mathcal{T}}(\lambda) = \partial_x - A(\cdot, \lambda), \qquad A(x, \lambda) = \begin{pmatrix} 0 & -1 \\ \alpha^{-1}(x)(\gamma(x) - \lambda) & \alpha^{-1}(x)\beta(x) \end{pmatrix},$$

which is obtained from rewriting $\tilde{\mathcal{L}}(\lambda) = 0$ into a first order ODE. Hence $A(x, \lambda)$ is a $(2N \times 2N)$-matrix. We write $\tilde{\mathcal{T}}_{L^2}(\lambda)$ and $\tilde{\mathcal{T}}_{\mathrm{BUC}}(\lambda)$ for the realization of $\tilde{\mathcal{T}}(\lambda)$ on $L^2(\mathbb{R}, \mathbb{C}^{2N})$ and $\mathrm{BUC}(\mathbb{R}, \mathbb{C}^{2N})$, respectively, with natural domains.

The following result is rather folklore, but does not seem to be explicitly stated in the literature. The equality of spectra for realizations on L^q, $1 \le q < \infty$, and the space C_0 of continuous functions vanishing at infinity follows from [52, Corollary 4.6]. For the more general theory of dichotomies and spectral mapping results on these spaces we refer to the monograph [12].

Proposition 3.2. *The following assertions are true, where $\lambda \in \mathbb{C}$:*

- *The spectrum, the point spectrum, and the essential spectrum of $\mathcal{L}_{H^1}$, $\mathcal{L}_{L^2}$, and $\mathcal{L}_{\mathrm{BUC}}$, respectively, all coincide.*
- *The operator $\mathcal{L}_{L^2} - \lambda$ is invertible if and only if $\tilde{\mathcal{T}}_{L^2}(\lambda)$ is invertible.*
- *The operator $\mathcal{L}_{L^2} - \lambda$ is Fredholm if and only if $\tilde{\mathcal{T}}_{L^2}(\lambda)$ is Fredholm. In this case the Fredholm indices coincide, as well as the dimension of the kernels.*

Proof. Lemma A.3 provides an isomorphism T from H^1 to L^2 and from H^3 to H^2 such that $\mathcal{L}_{H^1} = T^{-1}\mathcal{L}_{L^2}T$. Thus $\mathcal{L}_{H^1} - \lambda$ and $\mathcal{L}_{L^2} - \lambda$ have for each $\lambda \in \mathbb{C}$ the same invertibility and Fredholm properties. It remains to compare $\mathcal{L}_{L^2} - \lambda$ and $\mathcal{L}_{\mathrm{BUC}} - \lambda$. Since α is boundedly invertible, these operators have the same invertibility and Fredholm properties as $\tilde{\mathcal{L}}_{L^2}(\lambda)$ and $\tilde{\mathcal{L}}_{\mathrm{BUC}}(\lambda)$, respectively. It follows from [56, Theorem A.1] that their Fredholm properties are the same as those of $\tilde{\mathcal{T}}_{L^2}(\lambda)$ and $\tilde{\mathcal{T}}_{\mathrm{BUC}}(\lambda)$, respectively. It is further clear that the dimensions of the kernels coincide in both settings. Now in [10, Theorem 1.2] it is shown that the Fredholm properties of $\tilde{\mathcal{T}}_{L^2}(\lambda)$ are characterized by exponential dichotomies of the ODE $v' = A(\cdot, \lambda)v$ on both half-lines, and that in this case the dimension of the kernel of $\tilde{\mathcal{T}}_{L^2}(\lambda)$ depends only on the image of the dichotomies. This characterization is also true for $\tilde{\mathcal{T}}_{\mathrm{BUC}}(\lambda)$ with the same formula for the dimension of the kernel; see [47, Lemma 4.2] and [48]. Hence the invertibility and Fredholm properties of $\tilde{\mathcal{T}}_{L^2}(\lambda)$ and $\tilde{\mathcal{T}}_{\mathrm{BUC}}(\lambda)$ coincide, and if the operators are Fredholm, then the dimensions of the kernels coincide. This carries over to $\mathcal{L}_{L^2} - \lambda$ and $\mathcal{L}_{\mathrm{BUC}} - \lambda$ by the above considerations and shows the assertions. ∎

We finally remark that also for the realization of $\tilde{\mathcal{T}}(\lambda)$ on L^q with any $1 < q < \infty$ the Fredholm properties are characterized by exponential dichotomies (see [10, p. 94]). Putting this together with the arguments for [56, Theorem A.1], an appropriate generalization of

Lemma A.3, and interpolation, one obtains that the spectrum of $\mathcal{L}$ is independent of its realization on any of the spaces $H^{s,q}$ and $B^s_{q,r}$, where $s \geq 0$ and $1 \leq r \leq \infty$.

3.3. Computation of the spectrum. The invertibility and Fredholm properties of $\tilde{\mathcal{T}}(\lambda)$, and thus the characterization of the point and essential spectrum of $\mathcal{L}$, are described in terms of exponential dichotomies in [59, section 3.4]. This is independent of the variable leading order coefficients of $\mathcal{L}$ due to its quasi-linear origin and thus the same as for semilinear reaction-diffusion systems. We briefly describe the main points for each type of wave. A detailed discussion can also be found in [28, Chapter 3].

For a *homogeneous steady state* the point spectrum of the constant coefficient operator $\mathcal{L}$ is empty. Since the Fourier transform is an isomorphism on L^2, the (essential) spectrum can be determined by transforming $\mathcal{L}$ to

$$\widehat{\mathcal{L}}(\kappa) = -\alpha\kappa^2 + \mathrm{i}\beta\kappa + \gamma \in \mathbb{C}^{N\times N}, \qquad \kappa \in \mathbb{R}.$$

Now we have $\lambda \in \operatorname{spec}\mathcal{L}$ if and only if

$$d(\lambda,\kappa) := \det(\widehat{\mathcal{L}}(\kappa) - \lambda) = \det(A(\lambda) - \mathrm{i}\kappa) = 0$$

for some κ, which is called the dispersion relation for $\mathcal{L}$. The latter also means that $A(\lambda)$ is a nonhyperbolic matrix. Thus here it is straightforward to determine the spectrum, at least for N not too large.

For *pulses and fronts*, replacing the variable coefficients of $\mathcal{L}$ by their values at $\pm\infty$ leads to constant coefficient operators $\mathcal{L}^\pm$ whose spectrum is determined as just described. For pulses the essential spectrum of $\mathcal{L}$ already coincides with $\operatorname{spec}\mathcal{L}^\pm$. For fronts, $\operatorname{spec}\mathcal{L}^\pm$ equals the boundary of the essential spectrum of $\mathcal{L}$, which usually already determines stability issues. This is related to the fact that replacement by the values at infinity is a relatively compact perturbation of $\mathcal{L}$, which leaves Fredholm properties invariant (see [30, Theorem IV.5.26]). The point spectrum of a pulse or a front is determined by detecting intersections of the stable and unstable subspaces of $v' = A(\cdot,\lambda)v$. Here the *Evans function* [1, 23] is a powerful tool, and we refer readers to the survey [28, 59] and the references therein.

For a *wavetrain*, i.e., when $\overline{u}$ is periodic with wavelength (period) $L > 0$, the coefficients of $\mathcal{L}$ are periodic. The point spectrum is empty. Instead of the Fourier transform, here the Floquet–Bloch transform applies and yields (see [43, Theorem A.4], also for higher space dimensions)

$$\operatorname{spec}\mathcal{L} = \overline{\bigcup_{\kappa\in[0,2\pi/L)} \operatorname{spec}\hat{B}(\kappa)}. \tag{3.3}$$

For $\kappa \in [0, 2\pi/L)$ the operator $\hat{B}(\kappa) : H^2_{\mathrm{per}}(0,L) \subset L_{2,\mathrm{per}}(0,L) \to L_{2,\mathrm{per}}(0,L)$ with periodic boundary conditions is given by

$$\hat{B}(\kappa)U = \mathrm{e}^{-\mathrm{i}\kappa x}\mathcal{L}[\mathrm{e}^{\mathrm{i}\kappa x}U] = \widehat{\mathcal{L}}(\mathrm{i}\kappa + \partial_x)U,$$

where $\hat{\mathcal{L}}(\cdot)$ is the formal operator symbol of $\mathcal{L}$. Since $\operatorname{spec}\hat{B}(\kappa)$ consists only of eigenvalues, its spectrum is fully determined by the solvability of the family of boundary value problems

$$\hat{\mathcal{L}}(\mathrm{i}\kappa + \partial_x)U = \lambda U, \qquad U(0) = U(L).$$

In fact, also multiplicity of eigenvalues is determined via Jordan chains, as in [1, 59]. Notably, the spectrum again comes in curves, now an infinite countable union, since the eigenvalue problem for each κ still concerns an unbounded operator (rather than a matrix as in the case of a homogeneous steady state).

Via $V = \mathrm{e}^{\mathrm{i}\kappa x}U$, the boundary value problem formulation is equivalent to

$$\hat{\mathcal{L}}(\partial_x)V = \lambda V, \qquad V(0) = \mathrm{e}^{\mathrm{i}\kappa L}V(L).$$

By Floquet theory, this precisely means that the period map $\Pi(\lambda)$ of the evolution operator for the ODE $\hat{\mathcal{L}}(\partial_x)U = \lambda U$ possesses an eigenvalue (a Floquet multiplier) $\mathrm{e}^{\mathrm{i}\kappa L}$. Hence, also here a (linear) dispersion relation can be defined by

$$d(\lambda, \kappa) = \det\left(\Pi(\lambda) - \mathrm{e}^{\mathrm{i}\kappa L}\right) = 0,$$

which precisely characterizes the spectrum. An important difference from the case of homogeneous steady states is that $\lambda = 0$ always lies in the essential spectrum: x-independent coefficients of (3.1) yield a trivial zero Floquet exponent, which implies that $d(0, 0) = 0$. Indeed, $\hat{B}(0)\overline{u}_x = 0$ in this translation symmetric case.

Finally, in case of a *generalized wave train*, the boundary of the essential spectrum of $\mathcal{L}$ is, as above, obtained by replacing the coefficients of $\mathcal{L}$ with its periodic limits at $\pm\infty$ and considering the dispersion relation. The point spectrum is also given by an Evans-function; see [58, section 4] (there also the more general case of time periodic solutions, so-called defects, is treated).

4. Nonlinear stability and instability. For the nonlinearities a, f and a travelling wave solution $u_*(t, x) = \overline{u}(x - ct)$ of (1.1) we make the same assumptions as in the previous section. We consider (3.1),

$$u_t = (a(\overline{u} + u)u_x)_x + (a(\overline{u} + u)\overline{u}_x)_x + c(\overline{u}_x + u_x) + f(\overline{u} + u, \overline{u}_x + u_x),$$

in any of the well-posedness settings in a neighborhood of $S = \{\overline{u}(\cdot + \tau) - \overline{u} : \tau \in \mathbb{R}\}$.

4.1. Stability of pulses and fronts. Recall the precise notion of orbital stability from Definition 3.1. An application of [50, 51] gives the following conditional result. For more information on semisimple eigenvalues in Banach spaces, we refer the reader to [40, Appendix A.2].

Proposition 4.1. *Let $\overline{u}$ have constant asymptotic states. Assume that $\lambda = 0$ is a semisimple eigenvalue of $\mathcal{L}$ with eigenfunction $\overline{u}'$, i.e., $\ker\mathcal{L} = \operatorname{span}\{\overline{u}'\}$ and $X_0 = \ker\mathcal{L} \oplus \operatorname{im}\mathcal{L}$. Assume further that the remaining part of $\operatorname{spec}\mathcal{L}$ is strictly contained in $\{\operatorname{Re}\lambda < 0\}$. Then the travelling wave u_* is orbitally stable with asymptotic phase, and limit translates $u(\cdot + \tau_\infty)$ are approached exponentially.*

Proof. By translation invariance it suffices to consider S in a neighborhood of $\tau = 0$. The framework of Theorem 2.1 is that of [50, Theorem 2.1], provided that, in addition, A and F belong to C^1, which is guaranteed by the assumption on a and f. The setting of Theorem 2.6 is that of [51, Example 3]. To apply [50, Theorem 2.1] and [51, Theorem 3.1] it remains to verify that zero is normally stable, in the sense of [50, 51]. We have that S is a one-dimensional C^1-manifold, with tangent space at $\tau = 0$ spanned by $\overline{u}'$. By assumption, the tangent space coincides with the kernel of $\mathcal{L}$, and zero is a semisimple eigenvalue. Hence normal stability follows. ∎

For a quasi-linear variant of the Huxley equation, the above conditions have been verified in [50, section 5] by elementary arguments.

An abstract and more general variant of Proposition 4.1 and applications to semilinear problems can be found in [28, Chapter 4].

4.2. Instability of generalized pulses and fronts under localized perturbations. For localized perturbations, i.e., for $\mathcal{X} = H^2$ or $B^{2-2/p}_{2,p}$, a generalized pulse or front u_* is nonlinearly stable or unstable if the zero solution of (3.1) is stable or unstable, as a single equilibrium in the sense of Lyapunov. Nonlinear stability is a delicate issue (see the discussion in the introduction). In case of an unstable, spectral value we have the following.

Proposition 4.2. *If $\overline{u}$ has a periodic asymptotic state and* $\operatorname{spec}\mathcal{L} \cap \{\operatorname{Re}\lambda > 0\} \neq \emptyset$, *then the generalized front or pulse u_* is nonlinearly unstable with respect to localized perturbations from $\mathcal{X} = H^2$ or $\mathcal{X} = B^{2-2/p}_{2,p}$.*

Proof. The Lemmas A.1 and A.2 together with Proposition A.4 imply that the time-one solution map Φ_1 for (3.1) obtained in Theorems 2.4 and 2.5 from Theorem 2.1 is C^2 around zero, with $\Phi_1'(0) = e^{\mathcal{L}} \in \mathscr{L}(\mathcal{X})$. Considered on $\mathscr{L}(X_0)$, this operator has spectral radius larger than one by [40, Corollary 2.3.7]. Using $\mathcal{L} - \omega$ with sufficiently large $\omega > 0$ as a conjugate, this property carries over to $e^{\mathcal{L}}$ considered on $\mathscr{L}(X_1)$. Now it follows from interpolation that the realization of $e^{\mathcal{L}}$ on $\mathscr{L}(\mathcal{X})$ has spectral radius greater than one. Thus the zero solution of (3.1) is unstable by [26, Theorem 5.1.5]. ∎

4.3. Orbital instability. Without assuming a spectral gap or the existence of an unstable eigenvalue, we show that an unstable spectrum implies orbital instability.

Theorem 4.3. *The following assertions are true:*

- *Let $\overline{u}$ have constant asymptotic states. Assume* $\operatorname{spec}\mathcal{L} \cap \{\operatorname{Re}\lambda > 0\} \neq \emptyset$. *Then u_* is orbitally unstable with respect to localized and nonlocalized perturbations from $\mathcal{X} = H^2$, $B^{2-2/p}_{2,p}$, or $X = \mathrm{BUC}^2$.*
- *Let $\overline{u}$ have a periodic asymptotic state. Assume* $\operatorname{spec}\mathcal{L} \cap \{\operatorname{Re}\lambda > 0\} \neq \emptyset$. *Then u_* is orbitally unstable with respect to nonlocalized perturbations from $\mathcal{X} = \mathrm{BUC}^2$.*

This result is a direct consequence of the general orbital instability result, Theorem 4.5 below, for manifolds of equilibria: $\overline{u}' \in X_1$ in the settings under consideration, and $\mathcal{L}\overline{u}' = 0$ by the exponential convergence of $\overline{u}'$ at infinity and translation invariance of the equation.

The following lemma and its proof are generalizations of [26, Theorem 5.1.5] and [61]. Similar to those results, the proof establishes that perturbations of suitable approximate unstable eigenfunctions deviate from the manifold of equilibria.

Lemma 4.4. *Let X be a real Banach space, let $\mathcal{V} \subseteq X$ be an open neighborhood of zero, and let $\mathcal{E} \subset \mathcal{V}$ be an m-dimensional C^2-manifold containing zero. Let $\mathcal{E}$ be parametrized by an injective map $\psi : U \subset \mathbb{R}^m \to \mathcal{E}$ with $\psi(0) = 0$, where $\psi'(0)$ has full rank m. Assume that $T : \mathcal{V} \to X$ is continuous, that $T(u) = 0$ for $u \in \mathcal{E}$, and that there is an $M \in \mathscr{L}(X)$ with spectral radius greater than one such that, for some $\sigma > 1$,*

$$\|T(u) - Mu\| = \mathcal{O}(\|u\|^\sigma) \qquad \text{as } u \to 0. \tag{4.1}$$

Suppose further that $\partial_1\psi(0),\dots,\partial_m\psi(0) \in \ker(M - \mathrm{id})$. *Then* $u_* = 0$ *is orbitally unstable with respect to* $\mathcal{E}$ *under iterations of* T. *More precisely, there is an* $\varepsilon_0 > 0$ *such that for each* $\delta > 0$ *there are* $u_\delta \in \mathcal{V}$ *with* $\|u_\delta\| \le \delta$ *and* $N \in \mathbb{N}$ *such that* $T^n(u_\delta) \in \mathcal{V}$ *for* $n = 1,\dots,N$ *and* $\mathrm{dist}(T^N(u_\delta),\mathcal{E}) \ge \varepsilon_0$.

Proof. We will proceed in four steps.

Step 1. Let $\alpha_0, \beta > 0$ such that $B_{5\alpha_0}(0) \subset \mathcal{V}$ and

$$\|T(u_0) - Mu_0\| \le \beta\|u_0\|^\sigma, \qquad \|u_0\| \le 5\alpha_0. \tag{4.2}$$

There is an approximate eigenvalue $\lambda = re^{\mathrm{i}\theta}$ with $r > 1$ and $\theta \in \mathbb{R}$ in the spectrum of M. Furthermore, there are $\eta, K > 0$ with $r + \eta < r^\sigma$ and $\|M^n\| \le K(r+\eta)^n$ for all $n \ge 0$. In what follows we choose $\alpha \in (0, \alpha_0)$ stepwise possibly smaller and smaller, depending only on K, r, η, β, ψ.

Step 2. Let $\delta \in (0,\alpha)$ be given. As in the proof of [26, Lemma 5.1.4], we find $N \in \mathbb{N}$ such that

$$\frac{\alpha}{r^N} \le \delta, \qquad |\sin(N\theta)| \le \alpha, \tag{4.3}$$

and $u, v \in X$ with $\|u\| = 1$ and $\|v\| \le 1$ such that

$$\|M^n(u + \mathrm{i}v) - \lambda^n(u + \mathrm{i}v)\| \le \alpha, \qquad n = 1,\dots,N. \tag{4.4}$$

Here the norm is actually the complexified one, i.e., $\|w_1 + \mathrm{i}w_2\| = \|w_1\| + \|w_2\|$ for $w_1, w_2 \in X$.

Define $u_\delta := \frac{\alpha}{r^N} u \in X$ such that $\|u_\delta\| = \frac{\alpha}{r^N} \le \delta$. Let $n = 1,\dots,N$ be given. Assume inductively that $\|T^k(u_\delta)\| \le 5\alpha r^{k-N}$ for $k = 0,\dots,n-1$. Then $T^n(u_\delta)$ is well-defined and, as in the proof of [26, Theorem 5.1.5], we write

$$T^n(u_\delta) - \lambda^n u_\delta = \big(M^n u_\delta - \lambda^n u_\delta\big) + \sum_{k=0}^{n-1} M^{n-k-1}\big(T^{k+1}(u_\delta) - MT^k(u_\delta)\big). \tag{4.5}$$

Denote the right-hand side of this by $G_n + H_n$. We claim that

$$\|G_n\| \le \alpha^2 r^{-N} + 2\alpha|\sin(\theta n)|r^{n-N}, \qquad \|H_n\| \le C_M \alpha^\sigma r^{n-N}, \tag{4.6}$$

where $C_M = \frac{5^\sigma K\beta}{r^\sigma - r - \eta}$ is independent of n. To see this, we use (4.4) to obtain

$$\begin{aligned}\|G_n\| &\le \frac{\alpha}{r^N}\big(\|M^n u - (\operatorname{Re}\lambda^n)u + (\operatorname{Im}\lambda^n)\,v\| + \|(\operatorname{Im}\lambda^n)\,v\| + \|(\operatorname{Im}\lambda^n)\,u\|\big)\\ &\le \frac{\alpha}{r^N}\big(\|\operatorname{Re}((M^n - \lambda^n)(u + \mathrm{i}v))\| + 2r^n|\sin(\theta n)|\big)\\ &\le \alpha^2 r^{-N} + 2\alpha|\sin(\theta n)|r^{n-N}.\end{aligned} \tag{4.7}$$

For the sum H_n we use (4.2), that $\|T^k(u_\delta)\| \le 5\alpha r^{k-N} \le 5\alpha_0$ for $k \le n-1$, and that $r+\eta < r^\sigma$ to obtain

$$\begin{aligned}\|H_n\| &\le \sum_{k=0}^{n-1} K(r+\eta)^{n-k-1}\beta(5\alpha r^{k-N})^\sigma\\ &\le \alpha^\sigma 5^\sigma K\beta r^{\sigma(n-1-N)} \sum_{k=0}^{n-1}\left(\frac{r+\eta}{r^\sigma}\right)^{n-k-1} \le C_M\alpha^\sigma r^{n-N}.\end{aligned}$$

This shows the claim (4.6).

Now it follows from (4.5), (4.6), and $\sigma > 1$ that $\|T^n(u_\delta)\| \le 5\alpha r^{n-N}$, provided that $\alpha \le 1$ is such that $C_M \alpha^{\sigma-1} \le 1$. By induction, for all $n = 0, \dots, N$ we obtain that $T^n(u_\delta)$ is well-defined and the estimates $\|T^n(u_\delta)\| \le 5\alpha r^{n-N}$ and (4.6) hold true.

Step 3. As a consequence, for $\operatorname{dist}(T^N(u_\delta), \mathcal{E})$ we have to consider only $\zeta \in U$ such that $\|\psi(\zeta)\| \le 10\alpha$. Indeed, for $\|\psi(\zeta)\| > 10\alpha$ we have $\|T^N(u_\delta) - \psi(\zeta)\| > 5\alpha$, but $\|T^N(u_\delta) - \psi(0)\| = \|T^N(u_\delta))\| \le 5\alpha$. There is some small $\tau_0 > 0$ such that

$$\psi(\zeta) = \psi'(0)\zeta + \rho(\zeta), \qquad |\zeta| \le \tau_0, \tag{4.8}$$

where $\|\rho(\zeta)\| \le C_\rho |\zeta|^2$ for a constant C_ρ independent of $\zeta \in B_{\tau_0}(0)$. Since $\psi'(0)$ has full rank m, we have $C_{\psi'} = \min_{|\xi|=1} \|\psi'(0)\xi\| > 0$, and we can choose τ_0 such that $C_\rho \tau_0 \le C_{\psi'}/2$. Hence, with $\vartheta = 20/C_{\psi'}$ and small α, we obtain

$$\|\psi(\zeta)\| \ge \|\psi'(0)\zeta\| - C_\rho|\zeta|^2 > 10\alpha \qquad \text{for } \tau_0 \ge |\zeta| > \vartheta\alpha.$$

Then, with these choices,

$$\operatorname{dist}(T^N(u_\delta), \mathcal{E}) = \inf_{|\zeta| \le \vartheta\alpha} \|T^N(u_\delta) - \psi(\zeta)\|.$$

Step 4. Now let $|\zeta| \le \vartheta\alpha$. Then (4.5), (4.8), and the estimates (4.6) and $|\sin(N\theta)| \le \alpha$ yield

$$\begin{aligned} \|T^N(u_\delta) - \psi(\zeta)\| &\ge \|\lambda^N u_\delta - \psi'(0)\zeta\| - \|G_N\| - \|H_N\| - \|\rho(\zeta)\| \\ &\ge \|\alpha e^{\mathrm{i}N\theta} u - \psi'(0)\zeta\| - 3\alpha^2 - C_M\alpha^\sigma - \vartheta^2 C_{\psi''}\alpha^2. \end{aligned} \tag{4.9}$$

The vectors u and $\psi'(0)\zeta$ are linearly independent if α is sufficiently small. In fact, otherwise our assumption $\psi'(0)\zeta \in \ker(M - \mathrm{id})$ would imply that $Mu = u$. But, as in (4.7), the estimate (4.4) then yields $|\lambda - 1| = \|\lambda u - Mu\| \le \alpha^2 + 2\alpha$, which is impossible for small α.

We conclude that $\|e^{\mathrm{i}N\theta}u - \frac{1}{\alpha}\psi'(0)\zeta\|$ is bounded away from zero, uniformly for $|\zeta| \le \vartheta\alpha$. Hence, decreasing α once more if necessary, we obtain from (4.9) and $\sigma > 1$ that $\operatorname{dist}(T^N(u_\delta), \mathcal{E}) \ge \varepsilon_0$, where $\varepsilon_0 > 0$ is a multiple of α independent of δ. ■

Let us now apply the lemma to abstract quasi-linear problems:

$$\partial_t u = A(u)u + F(u), \quad t > 0, \qquad u(0) = u_0. \tag{4.10}$$

We denote by $\mathcal{L}(u_*) = A(u_*) + A'(u_*)[\cdot, u_*] + F'(u_*)$ the linearization of the right-hand side at u_*.

Theorem 4.5. *Assume the setting of either Theorem 2.1 or 2.6, and in addition that A and F are C^2. Let $\mathcal{E} \subset \mathcal{V} \cap X_1$ be an m-dimensional C^2-manifold of equilibria of (4.10), parametrized by $\psi : U \subset \mathbb{R}^m \to \mathcal{E}$, and let $u_* \in \mathcal{E}$ satisfy*

- $\operatorname{spec} \mathcal{L}(u_*) \cap \{\operatorname{Re} \lambda > 0\} \ne \emptyset$,
- $\partial_1\psi(\zeta_*), \dots, \partial_m\psi(\zeta_*) \in \ker \mathcal{L}(u_*)$ *for* $u_* = \psi(\zeta_*)$.

Then u_* is orbitally unstable in $\mathcal{V} \subseteq \mathcal{X}$ with respect to $\mathcal{E}$.

Proof. Shrink $\mathcal{V}$ around u_* if necessary such that $t^+(u_0) \ge 1$ for each $u_0 \in \mathcal{V}$. Let $\Phi_1 : V \to \mathcal{X}$ be the time-one solution map for (4.10). Define $T(u_0) = \Phi_1(u_* + u_0) - (u_* + u_0)$

for u_0 close to u_*. Then T is continuous, $T(u) = 0$ for $u \in \mathcal{E} \cap \mathcal{V}$, and T satisfies (4.1) with $M = e^{\mathcal{L}(u_*)} \in \mathscr{L}(\mathcal{X})$, as a consequence of Proposition A.4 for the setting of Theorem 2.1, and of [42, Proposition 6.2] for the setting of Theorem 2.6. Moreover, M has spectral radius larger than one by [40, Corollary 2.3.7] and interpolation, and $\partial_j \psi(\zeta_*) \in \ker(M - \mathrm{id})$ follows from the assumption. Thus Lemma 4.4 applies. ■

Of course, Lemma 4.4 applies in any well-posedness setting for nonlinear parabolic problems.

5. A generalized Gray–Scott–Klausmeier model. For illustration of the previous results, let us consider the model (1.2) for water-vegetation interaction in semiarid landscapes:

$$\begin{aligned} w_t &= (w^2)_{xx} + Cw_x + A(1-w) - wv^2, \\ v_t &= Dv_{xx} - Bv + wv^2. \end{aligned} \tag{5.1}$$

Here A is roughly a measure of the rainfall. On the one hand, (5.1) is (a rescaling of) the Klausmeier model for banded vegetation patterns on a sloped terrain from [32], when removing the porous medium term $(w^2)_{xx}$. On the other hand, upon replacing $(w^2)_{xx}$ by w_{xx} and setting $C = 0$, (5.1) is precisely the semilinear Gray–Scott model, which has been extensively studied in the past decades; see, e.g., [16,18,44] and the references therein. The relations between these different models in terms of periodic patterns have been studied in [65]. From an application point of view it is important to know in which patterned state these model systems may reside, and thus to establish well-posedness as well as existence, stability, and instability of patterns.

In order to illustrate the straightforward applicability of the frameworks of the previous sections, we show well-posedness around travelling waves with first component bounded away from zero. We then consider homogeneous steady states and wavetrains and derive the dispersion relations. These are illustrated by numerical computations of spectra when passing a Turing–Hopf bifurcation and a sideband instability.

5.1. Well-posedness for perturbations of travelling waves. To cast (5.1) into the form (1.1) we set $\mathbf{u} = (w, v)$ and define the smooth nonlinearities $a : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$a(\mathbf{u}) = \begin{pmatrix} 2w & 0 \\ 0 & D \end{pmatrix}, \qquad f(\mathbf{u}, \mathbf{u}_x) = \begin{pmatrix} Cw_x + A(1-w) - wv^2 \\ -Bv + wv^2 \end{pmatrix}.$$

Then (5.1) is equivalent to

$$\mathbf{u}_t = (a(\mathbf{u})\mathbf{u}_x)_x + f(\mathbf{u}, \mathbf{u}_x).$$

We see that $a(\mathbf{u})$ is positive definite only for $w > 0$, and thus (5.1) fails to be parabolic for $w \le 0$. We therefore restrict our analysis to $w > 0$. From the quasi-positive structure of f for $A > 0$ and the smoothness of solutions given by the well-posedness, it readily follows that (5.1) preserves $w > 0$ on the maximal existence interval.

Assume that $\mathbf{u}_*(t, x) = \overline{\mathbf{u}}(x - ct)$ is a travelling wave solution of (5.1) with profile

$$\overline{\mathbf{u}} = (\overline{w}, \overline{v}) \in \mathrm{BC}^\infty(\mathbb{R}, \mathbb{R}^2),$$

satisfying $\overline{w} \ge \delta > 0$, and speed $c \in \mathbb{R}$. Note that this includes homogeneous steady states. Denote the co-moving frame $x - ct$ again by x. As for (3.1), the evolution of perturbations $\mathbf{u}$

of $\overline{\mathbf{u}}$ under (5.1) is governed by

$$\mathbf{u}_t = (a(\overline{\mathbf{u}}+\mathbf{u})\mathbf{u}_x)_x + (a(\overline{\mathbf{u}}+\mathbf{u})\overline{\mathbf{u}}_x)_x + c(\overline{\mathbf{u}}_x+\mathbf{u}_x) + f(\overline{\mathbf{u}}+\mathbf{u}, \overline{\mathbf{u}}_x+\mathbf{u}_x). \tag{5.2}$$

Choose $\mathcal{V}$ as any open subset of $\mathcal{X} = H^2$, $\mathcal{X} = B^{2-2/p}_{2,p}$ with $p > 2$ sufficiently large, or $\mathcal{X} = \mathrm{BUC}$ such that $\overline{w}+w$ is positive and bounded away from zero for all $\mathbf{u} = (w,v) \in \mathcal{V}$. This is possible in view of the Sobolev embeddings $H^2 \subset \mathrm{BUC}$ and (2.7). Theorems 2.4, 2.5, or 2.7 apply, and each yields local well-posedness of (5.2) in the corresponding choice of $\mathcal{V}$ and in the sense of Theorems 2.1 and 2.7. Solutions are in fact smooth in space and time (see Remark 2.3).

The eigenvalue problem for the linearization of the right-hand side of (5.2) in $\mathbf{u} = 0$ is for $\lambda \in \mathbb{C}$ given by

$$\begin{aligned} \lambda w &= 2\overline{w}w_{xx} + 4\overline{w}_x w_x + 2\overline{w}_{xx}w + (C+c)w_x - Aw - \overline{v}^2 w - 2\overline{w}\,\overline{v}v, \\ \lambda v &= Dv_{xx} + cv_x - Bv + \overline{v}^2 w + 2\overline{w}\,\overline{v}v. \end{aligned} \tag{5.3}$$

By Proposition 3.2, the spectrum of the linearization is independent of the above functional analytical frameworks. A brief account for the computation of the spectrum is given in section 3.3, and we refer to [59] for a survey. Nonlinear stability or instability of $\mathbf{u}_*$ can be deduced from the results in section 4 in some situations, as pointed out below.

5.2. Homogeneous steady states. These are solutions $w(t,x) = w_*$, $v(t,x) = v_* \in \mathbb{R}$ to (5.1) that are time- and space-independent and thus solve the algebraic equations arising from vanishing space and time derivatives. We readily compute that the possibilities are $(w_0, v_0) = (1, 0)$ and, in case $A \geq 4B^2$,

$$w_\pm = \frac{1}{2A}\left(A \mp \sqrt{A^2 - 4AB^2}\right), \qquad v_\pm = \frac{1}{2B}\left(A \pm \sqrt{A^2 - 4AB^2}\right).$$

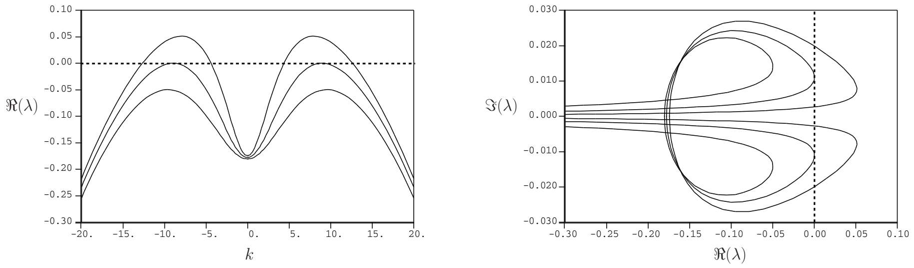
The state (w_0, v_0), with zero vegetation, represents the desert (even though there is nonzero "water"), while the equilibria (w_+, v_+) and (w_-, v_-) represent co-existing homogeneously vegetated states. At $A = A_{\mathrm{sn}} = 4B^2$, the latter two collapse in a saddle-node bifurcation. The spectrum of the linearization in (w_*, v_*) can be computed from the usual dispersion relation $d(\lambda, \kappa) = 0$, where

$$d(\lambda,\kappa) = \det \begin{pmatrix} -2w_*\kappa^2 + \mathrm{i}\kappa(C+c) - A - v_*^2 - \lambda & -2w_* v_* \\ v_*^2 & -D\kappa^2 + \mathrm{i}\kappa c - B + 2w_* v_* - \lambda \end{pmatrix}$$

is obtained from a Fourier transform; see section 3.3.

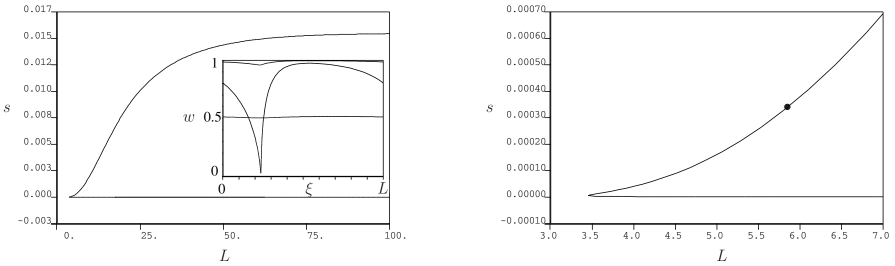
An origin of patterns is a (supercritical) Turing–Hopf bifurcation of the steady state (w_+, v_+) that occurs as A decreases from larger values, as shown in [65]. It is in fact straightforward to study bifurcations of spatially periodic travelling waves, as this involves only ODE analysis. As a side note on Turing–Hopf bifurcations, we mention that the dynamics of (5.1) near onset is formally approximated by a complex Ginzburg–Landau equation (see [65]), but a rigorous justification has not been established for quasi-linear problems, to our knowledge.

In order to locate the Turing–Hopf bifurcation, we need to study the spectrum of the linearization in (w_+, v_+). For illustration, in Figure 1 we plot the spectrum obtained



(a) (b)

Figure 1. *Spectra of the homogeneous steady state* (w^+, v^+) *of* (5.1) *for* $B = C = 0.2$, $D = 0.001$ *before the Turing–Hopf instability,* $A = 0.63$ *(stable); near to it,* $A = 0.53$*; and after it,* $A = 0.43$ *(unstable).* (a) *Real part of the spectrum versus linear wavenumber.* (b) *Imaginary part of the spectrum versus real part.*



(a) (b)

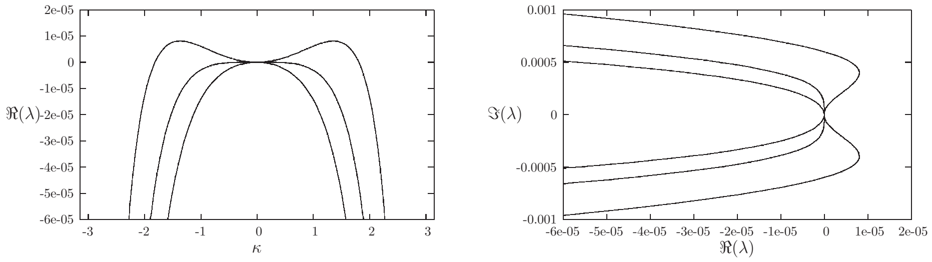
Figure 2. (a) *Sample bifurcation diagram of wavetrains for* $A = 0.02$, $B = C = 0.2$, $D = 0.001$. *At* $L \approx 3.45$ *a fold occurs, and both branches appear to terminate in a homoclinic bifurcation as* $L \to \infty$. *The inset shows profiles of solutions at the fold* ($w \approx 0.5$) *and near* $L = 80$ *on the upper and lower* ($w \approx 1$) *branches.* (b) *Magnification of the bifurcation diagram with bullet marking the location of the sideband instability at* $L \approx 5.98$. *Solutions on the branch for increasing period are spectrally stable.*

numerically (using AUTO [17]) from the dispersion relation as the parameter A passes through the aforementioned Turing–Hopf bifurcation. Since the spectrum is unstable after passing the Turing–Hopf instability (e.g., $A = 0.43$ in Figure 1), the steady state is expected to be unstable under the nonlinear evolution. Indeed, this is the case thanks to Theorem 4.3.

5.3. Wavetrains. The patterns emerging at the Turing–Hopf bifurcation are periodic wavetrains, which are solutions to (5.1) of the form

$$(w_*, v_*)(t, x) = (\tilde{w}, \tilde{v})(kx - \omega t),$$

with a 2π-periodic profile $(\tilde{w}, \tilde{v})$. Here ω is called the frequency and k the wavenumber. As noted in [65], the existence region of wavetrains to (5.1) in parameter space extends far from the Turing–Hopf bifurcation and even beyond the saddle-node bifurcation $A = A_{\mathrm{sn}}$ of homogeneous equilibria with vegetation. In Figure 2 we plot a branch of wavetrain solutions for $A < A_{\mathrm{sn}}$ that appears to terminate in another type of travelling wave: pulses, which are spatially homoclinic orbits.



(a) (b)

Figure 3. *Spectra of the wavetrains for $B = C = 0.2$, $D = 0.001$, $A = 0.02$ before the sideband instability, $L = 5.9$ (stable); near to it, $L \approx 5.98$; and after it, $L = 6.1$ (unstable).* (a) *Real part versus linear wavenumber.* (b) *Imaginary part versus real part.*

In order to link to the formulations for travelling waves, let us cast wavetrains as equilibria $(w_*, v_*)(t, x) = (\overline{w}, \overline{v})(x - ct)$ in the co-moving frame $x - ct$ with speed $c = \frac{\omega}{k}$. The eigenvalue problem of the linearization of (5.1) in a wavetrain is then given by (5.3), with coefficients of period $L = 2\pi/k$ stemming from $(\overline{w}, \overline{v})$.

The approach via Fourier transform is less useful, because the linearization is not diagonal in Fourier space due to the x-dependent coefficients. As a substitute, one uses the Floquet–Bloch transform, which replaces the eigenvalue problem on $\mathbb{R}$ by a family of eigenvalue problems on the wavelength interval $[0, L]$ (see section 3.3). Specifically, this can be cast as the family of boundary value problems for $\kappa \in [0, 2\pi)$ given by (5.3), with ∂_x replaced by $\partial_x + \mathrm{i}\kappa/L$ and L-periodic boundary conditions.

With a curve of the spectrum of a wavetrain connected to the origin $\lambda = 0$ (due to translation symmetry), a change in its curvature is a typical destabilization upon parameter variation. This so-called *sideband instability* is illustrated in Figure 3, where we plot spectra of wavetrains in (5.1) passing through a sideband instability as the wavelength L changes. For these computations, we implemented the first order formulation of the dispersion relation numerically in Auto based on the algorithm from [53].

As for the homogeneous steady state, the wavetrains with unstable spectrum (e.g., $L = 6.1$ in Figure 3) are expected to be (orbitally) unstable under the nonlinear evolution of (5.1); see Proposition 4.2 and Theorem 4.3.

Appendix A. Auxiliary results.

A.1. Superposition operators. We give some details for the properties of the nonlinear maps employed in the well-posedness results.

Lemma A.1. *Let $U_1, U_2 \subset \mathbb{R}^N$ be open neighborhoods of zero, let $a : \mathbb{R} \times U_1 \to \mathbb{R}^N$ be C^{k+3}, and let $f : \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}^N$ be C^{k+2}, with $f(\cdot, 0, 0) \in H^1$. Define the superposition operators*

$$A(u)v = (a(\cdot, u)v_x)_x, \qquad F(u) = f(\cdot, u, u_x).$$

Then there is an open subset $\mathcal{V}$ of H^2 such that $A \in C^k(\mathcal{V}, \mathscr{L}(H^3, H^1))$ and $F \in C^k(\mathcal{V}, H^1)$, and both maps are Lipschitz on bounded subsets of $\mathcal{V}$. One can take $\mathcal{V} = H^2$ if $U_1 = U_2 = \mathbb{R}^N$. At $\overline{u} \in \mathcal{V}$, the derivatives for $u \in H^2$ and $v \in H^3$ are given by

$$A'(\overline{u})[u, v] = (\partial_2 a(\cdot, \overline{u})[u, v_x])_x, \qquad F'(u_0)v = \partial_2 f(\cdot, \overline{u}, \overline{u}_x)v + \partial_3 f(\cdot, \overline{u}, \overline{u}_x)v_x.$$

Proof. Choose $\mathcal{V} \subseteq H^2$ such that for $u \in \mathcal{V}$ the closure of the images of $u, u_x \in H^1 \subset \mathrm{BC}$ are uniformly contained in U_1 and U_2, respectively. Let $u \in \mathcal{V}$. For $h \in H^2$ we use $\|uh\|_{L^2} \leq \|u\|_{\mathrm{BC}}\|h\|_{L^2}$ and $\|u\|_{\mathrm{BC}} \leq C\|u\|_{H^1}$ to estimate

$$\begin{aligned}\|\partial_2 f(\cdot, u, u_x)h\|_{H^1} &\leq \|\partial_2 f(\cdot, u, u_x)\|_{\mathrm{BC}}(\|h\|_{L^2} + \|h_x\|_{L^2})\\ &\quad + \|f''(\cdot, u, u_x)\|_{\mathrm{BC}}(\|h\|_{L^2} + \|u_x\|_{L^2}\|h\|_{\mathrm{BC}} + \|u_{xx}\|_{L^2}\|h\|_{\mathrm{BC}})\\ &\leq C(\|f'(\cdot, u, u_x)\|_{\mathrm{BC}} + \|f''(\cdot, u, u_x)\|_{\mathrm{BC}}\|u\|_{H^2})\|h\|_{H^1}.\end{aligned}$$

In the same way we obtain

$$\|\partial_3 f(\cdot, u, u_x)h_x\|_{H^1} \leq C(\|f'(\cdot, u, u_x)\|_{\mathrm{BC}} + \|f''(\cdot, u, u_x)\|_{\mathrm{BC}}\|u\|_{H^2})\|h\|_{H^2}.$$

Defining $F'(u)h = \partial_2 f(\cdot, u, u_x)h + \partial_3 f(\cdot, u, u_x)h_x$, we thus have $F'(u) \in \mathscr{L}(H^2, H^1)$ and that $u \mapsto F'(u)$ is bounded on bounded subsets of $\mathcal{V}$. If h is small, then the pointwise identity

$$\begin{aligned}&F(u + h) - F(u) - F'(u)h\\ &\quad = \int_0^1 \int_0^1 \big(\partial_{22} f(\cdot, u + \tau s h, u_x)[h, \tau h] + \partial_{33} f(\cdot, u, u_x + \tau s h_x)[h_x, \tau h_x]\big)\mathrm{d}\tau \mathrm{d}s,\end{aligned}$$

and the same types of estimates as above yield

$$\|F(u + h) - F(u) - F'(u)h\|_{H^1} \leq C(f, h)\|h\|_{H^2}^2,$$

where $C(f, h)$ is bounded as $h \to 0$. These arguments and $f(\cdot, 0, 0) \in H^1$ yield $F(u) \in H^1$ for $u \in \mathcal{V}$ and the differentiability of F in $\mathcal{V}$. The Lipschitz property follows from the boundedness of F'. Iteration for higher derivatives gives $F \in C^k$. The arguments apply to $u \mapsto a(u)$ on H^2 as well, which yields the assertion on A. ■

Note that if f is independent of u_x, then the arguments from the proof above show that $f : H^1 \to H^1$ is smooth.

Lemma A.2. *In the situation of Theorem 2.5, assume in addition that a and f are C^{k+2} for some $k \geq 0$. Let A and F be defined by*

$$A(u)v = \partial_i(a_{ij}(\overline{u} + u)\partial_j v), \quad F(u) = \partial_i(a_{ij}(\overline{u} + u)\partial_j \overline{u}) + c_i\partial_i(\bar{u} + u) + f(\overline{u} + u).$$

Then for all sufficiently large $p > 2$ there is is an open neighborhood $\mathcal{V} \subset B_{2,p}^{2-2/p}$ of the zero function such that $F \in C^k(\mathcal{V}, L^2)$ and $A \in C^k(\mathcal{V}, \mathscr{L}(H^2, L^2))$, and both maps are Lipschitz on bounded sets. One can take $\mathcal{V} = B_{2,p}^{2-2/p}$ if $U = \mathbb{R}^N$.

Proof. Since $n \leq 3$, from Sobolev's embedding (2.7) we find $p > 2$ such that $B_{2,p}^{2-2/p} \subset H^{1,4} \cap \mathrm{BC}$. Then $\mathcal{V}$ can be chosen such that the image of $\overline{u} + u$ is strictly contained in U, uniformly in $u \in \mathcal{V}$. The regularity of A and F can be derived as in Lemma A.1, using $F(0) = 0$. The need for $B_{2,p}^{2-2/p} \subset H^{1,4}$ and thus also $H^2 \subset H^{1,4}$ comes from the nonlinear gradient terms. Indeed, assume for simplicity that $\overline{u} = 0$. Then for $u_1, u_2 \in B_{2,p}^{2-2/p}$ and $v \in H^2$ we can estimate

$$\begin{aligned}\|a_{ij}'(u_1)\partial_i u_1\partial_j v - a_{ij}'(u_2)\partial_i u_2\partial_j v\|_{L^2} &\leq \|a_{ij}'(u_1)\partial_i u_1 - a_{ij}'(u_2)\partial_i u_2\|_{L^4}\|\partial_j v\|_{L^4}\\ &\leq \big(\|a_{ij}'(u_1)\|_{\mathrm{BC}}\|u_1 - u_2\|_{H^{1,4}} + \|u_2\|_{H^{1,4}}\|\|a_{ij}'(u_1) - a_{ij}'(u_2)\|_{\mathrm{BC}}\big)\|v\|_{H^{1,4}},\end{aligned}$$

employing Hölder's inequality $L^4 \cdot L^4 \subset L^2$ in the first line. ■

A.2. A commuting isomorphism for elliptic operators. The following auxiliary result for second order differential operators allows us to transfer spectral properties from L^2 to H^1 by conjugation.

Lemma A.3. *Let $\alpha, \beta, \gamma \in \mathrm{BC}^1(\mathbb{R}, \mathbb{R}^{N\times N})$, and assume that $\alpha(x)$ is positive definite, uniformly in x. Then there is a continuous isomorphism $T : H^1 \to L^2$, which also maps $T : H^3 \to H^2$ isomorphically, that commutes on H^3 with the operator $\varphi \mapsto \mathcal{L}\varphi := \alpha\varphi_{xx} + \beta\varphi_x + \gamma\varphi$.*

Proof. The isomorphism T will be the square root of a shift of $\mathcal{L}$. The main point is to show that its domain for the realization on H^2 is H^3.

Denote by $\mathcal{L}_{L^2}$ the realization of $\mathcal{L}$ on L^2, with domain H^2. The properties of α together with [6, Theorem 9.6] imply that there is some $\omega > 0$ such that $B = \omega - \mathcal{L}_{L^2}$ is a (negative) sectorial operator and has a bounded holomorphic functional calculus of angle strictly smaller than $\frac{\pi}{2}$. In particular, $T := B^{1/2}$ is a well-defined continuous isomorphism $D(B^{1/2}) \to L^2$; see [66, Theorem 1.15.2]. The boundedness of the holomorphic calculus of B implies that it has the property of bounded imaginary powers. Therefore, combining [39, Lemma 4.1.11] with [66, Theorem 1.15.3] (or [39, Theorem 4.2.6]), we have $D(B^{1/2}) = [L^2, H^2]_{1/2}$, where $[\cdot,\cdot]_{1/2}$ denotes complex interpolation (see [11,39,66]). Since $[L^2, H^2]_{1/2} = H^1$ by [66, Remark 2.4.2/2], it follows that $T : H^1 \to L^2$ is an isomorphism.

Next, we show that $T : H^3 \to H^2$ is an isomorphism. Again by [66, Theorem 1.15.2], T also maps isomorphically $D(B^{3/2}) \to D(B) = H^2$. We show that $D(B^{3/2}) = H^3$ as Banach spaces. By [39, Lemma 4.1.16 and Theorem 4.1.11] and the previous considerations, we have

$$D(B^{3/2}) = \{u \in D(B) : Bu \in D(B^{1/2})\} = \{u \in H^2 : \mathcal{L}u \in H^1\}.$$

For $u \in H^3$ we clearly have $\mathcal{L}u \in H^1$, and hence $H^3 \subseteq D(B^{3/2})$. Conversely, let $u \in H^2$ such that $\mathcal{L}u \in H^1$. Then $\alpha u_{xx} = \psi := -\beta u_x - \gamma u + \mathcal{L}u \in H^3$. By assumption, the coefficient α is pointwise invertible, with $\alpha^{-1} \in \mathrm{BC}^1$. Therefore $u_{xx} = \alpha^{-1}\psi \in H^1$, and so $u \in H^3$. We conclude that $D(B^{3/2}) = H^3$ as sets. Arguing as before, we get

$$\|u\|_{D(B^{3/2})} = \|u\|_{H^2} + \|\mathcal{L}u\|_{H^1} \le C\|u\|_{H^3}$$

for a constant C that is independent of u. Since we already know that H^3 is complete with respect to $\|\cdot\|_{D(B^{3/2})}$ and $\|\cdot\|_{H^3}$, the converse estimate follows from the open mapping theorem.

Finally, it follows from [39, Theorem 4.1.6] that $\omega - \mathcal{L}_{L^2}$ and its square root T commute on H^3. This implies that also $\mathcal{L}_{L^2}$ commutes with T. ■

The assertion of the above lemma remains valid, with literally the same proof, if one replaces the L^2 setting by an L^q setting, where $q \in (1, \infty)$.

A.3. The time-one solution map. We use the implicit function theorem to prove that in the neighborhood of an equilibrium the solution semiflow obtained from Theorem 2.1 for (2.1) is as smooth as the right-hand side. See [26, Theorem 3.4.4] for the semilinear case, as well as [40, Theorem 8.3.4] and [5, Theorem 4.1] for quasi-linear frameworks.

Proposition A.4. *In the situation of Theorem 2.1, assume additionally that*

$$A \in C^k(\mathcal{V}, \mathscr{L}(X_1, X_0)), \qquad F \in C^k(\mathcal{V}, X_0)$$

for some $k \in \mathbb{N}$. Let $u_* \in \mathcal{V} \cap X_1$ be an equilibrium of (2.1); i.e., $A(u_*)u_* + F(u_*) = 0$. Then for any $\tau > 0$ there is a neighborhood $\mathcal{U} \subseteq \mathcal{V}$ of u_* such that the time-τ map $u_0 \mapsto \Phi_\tau(u_0) =$

$u(\tau; u_0)$ for the solution semiflow for (2.1) is well-defined and belongs to $C^k(\mathcal{U}, \mathcal{X})$. Moreover, let $\mathcal{L}_* = A(u_*) + A'(u_*)[\cdot, u_*] + F'(u_*)$. Then $\Phi'_\tau(u_*) = e^{\tau \mathcal{L}_*}$.

Proof. We assume $\mathcal{V} = \mathcal{X}$. Set $\mathbb{E}_1 = H^{1,p}(0,\tau; X_0) \cap L^p(0,\tau; X_1)$ and $\mathbb{E}_0 = L^p(0,\tau; X_0)$, and consider

$$\Psi : \mathbb{E}_1 \times \mathcal{X} \to \mathbb{E}_0 \times \mathcal{X}, \qquad \Psi(u, u_0) = (\partial_t u - A(u)u - F(u), u(0) - u_0).$$

Note that $u \in \mathbb{E}_1$ solves (2.1) on $(0,\tau)$ with initial value $u_0 \in \mathcal{X}$ if and only if $\Psi(u, u_0) = (0,0)$. Consider u_* as an element of $\mathbb{E}_1$. Then $\Psi(u_*, u_*) = (0,0)$. The assumptions on A and F imply $\Psi \in C^k(\mathbb{E}_1 \times \mathcal{X}, \mathbb{E}_0 \times \mathcal{X})$ and

$$D_1\Psi(u_*, u_*)v = (\partial_t v - \mathcal{L}_* v, v(0)), \qquad v \in \mathbb{E}_1.$$

From the proof of Theorem 2.1 we know that $-A(u_*)$ enjoys maximal L^p-regularity. The linear operator $A'(u_*)[\cdot, u_*] + F'(u_*)$ is continuous from $\mathcal{X} = (X_0, X_1)_{1-1/p,p}$ to X_0; i.e., it is of lower order. Thus $-\mathcal{L}_*$ has maximal L^p-regularity as well; see [20, Theorem 6.2]. In other words, $D_1\Psi(u_*, u_*) \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0 \times \mathcal{X})$ is an isomorphism. This gives a neighborhood $\mathcal{U}$ of u_* in $\mathcal{X}$ such that $u_0 \mapsto u(\cdot; u_0)$ belongs to $C^k(\mathcal{U}, \mathbb{E}_1)$, where $u(\cdot; u_0)$ is the solution of (2.1) on $(0,\tau)$. Moreover, for $v_0 \in \mathcal{X}$ we differentiate $\Psi(u(\cdot; u_0), u_0) = 0$ in u_* to get that

$$D_{u_0} u(\cdot; u_*) v_0 = -D_1\Psi(u_*, u_*)^{-1} D_2\Psi(u_*, u_*) v_0 = -D_1\Psi(u_*, u_*)^{-1}(0, -v_0)$$

is the unique solution $v \in \mathbb{E}_1$ of $\partial_t v - \mathcal{L}_* v = 0$ on $(0,\tau)$ with $v(0) = v_0$, i.e., $D_{u_0} u(\cdot; u_*) = e^{\cdot \mathcal{L}_*}$. Finally, the trace at time τ is linear and continuous as a map $\mathbb{E}_1 \to \mathcal{X}$; see [66, Theorem 1.14.5]. Applying this to $u(\cdot; u_0)$ gives the assertion for Φ_τ. ■

Acknowledgments. M.M. and E.S. thank the CWI for its kind hospitality. The authors thank Johannes Höwing for his comments and are grateful to the anonymous referees for their valuable hints and for pointing out additional references.

REFERENCES

[1] J. Alexander, R. Gardner, and C. K. R. T. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. Reine Angew. Math., 410 (1990), pp. 167–212.
[2] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in Function Spaces, Differential Operators and Nonlinear Analysis, H.-J. Schmeisser and H. Triebel, eds., Teubner-Texte Math. 133, Teubner, Stuttgart, 1993, pp. 9–126.
[3] H. Amann, *Linear and Quasilinear Parabolic Problems. Volume* I: *Abstract Linear Theory*, Birkhäuser, Basel, 1995.
[4] H. Amann, *Nonlocal quasilinear parabolic equations*, Russian Math. Surveys, 60 (2005), pp. 1021–1035.
[5] H. Amann, *Quasilinear parabolic problems via maximal regularity*, Adv. Differential Equations, 10 (2005), pp. 1081–1110.
[6] H. Amann, M. Hieber, and G. Simonett, *Bounded H_∞-calculus for elliptic operators*, Differential Integral Equations, 7 (1994), pp. 613–653.
[7] S. B. Angenent, *Nonlinear analytic semiflows*, Proc. R. Soc. Edinburgh A, 115 (1990), pp. 91–107.
[8] J. T. Beale, *Large-time regularity of viscous surface waves*, Arch. Ration. Mech. Anal., 84 (1984), pp. 307–352.
[9] M. Beck, B. Sandstede, and K. Zumbrun, *Nonlinear stability of time-periodic shock waves*, Arch. Ration. Mech. Anal., 196 (2010), pp. 1011–1076.

[10] A. Ben-Artzi and I. Gohberg, *Dichotomy of systems and invertibility of linear ordinary differential operators*, Oper. Theory Adv. Appl., 56 (1992), pp. 90–119.

[11] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer, New York, 1976.

[12] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys Monogr. 70, AMS, Providence, RI, 1999.

[13] J. Cholewa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, UK, 2000.

[14] P. Clément and S. Li, *Abstract parabolic quasilinear equations and applications to a groundwater flow problem*, Adv. Math. Sci. Appl., 3 (1994), pp. 17–32.

[15] P. Clément and G. Simonett, *Maximal regularity in continuous interpolation spaces and quasilinear parabolic problems*, J. Evol. Equ., 1 (2001), pp. 39–67.

[16] W. Chen and M. J. Ward, *Oscillatory instabilities and dynamics of multi-spike patterns for the one-dimensional Gray-Scott model*, European J. Appl. Math., 20 (2009), pp. 187–214.

[17] E. J. Doedel, *AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations*, software, 2011, available online at http://cmvl.cs.concordia.ca/auto.

[18] A. Doelman, T. J. Kaper, and P. Zegeling, *Pattern formation in the one-dimensional Gray-Scott model*, Nonlinearity, 10 (1997), pp. 523–563.

[19] A. Doelman, B. Sandstede, A. Scheel, and G. Schneider, *The dynamics of modulated wave trains*, Mem. Amer. Math. Soc., 199 (2009).

[20] G. Dore, *Maximal regularity for abstract Cauchy problems*, Adv. Differential Equations, 5 (2000), pp. 293–322.

[21] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[22] J. Escher, J. Prüss, and G. Simonett, *A new approach to the regularity of solutions for parabolic equations*, in Evolution Equations: Proceedings in Honor of J. A. Goldstein's 60th Birthday, Marcel Dekker, New York, 2003, pp. 167–190.

[23] J. W. Evans, *Nerve axon equations.* IV. *The stable and the unstable impulse*, Indiana Univ. Math. J., 24 (1974/75), pp. 1169–1190.

[24] J. A. Griepentrog and L. Recke, *Local existence, uniqueness, and smooth dependence for nonsmooth quasilinear parabolic problems*, J. Evol. Equ., 10 (2010), pp. 341–375.

[25] T. Häcker, G. Schneider, and H. Uecker, *Self-similar decay to the marginally stable ground state in a model for film flow over inclined wavy bottoms*, Electron. J. Differential Equations, 2012 (2012), 61.

[26] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer, Heidelberg, Berlin, New York, 1981.

[27] M. Holzer, A. Doelman, and T. J. Kaper, *Existence and stability of traveling pulses in a reaction-diffusion-mechanics system*, J. Nonlinear Sci., 23 (2013), pp. 129–177.

[28] T. Kapitula and K. Promislow, *Spectral and Dynamical Stability of Nonlinear Waves*, Appl. Math. Sci. 185, Springer, New York, 2013.

[29] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, in Spectral Theory and Differential Equations, Lecture Notes in Math. 448, Springer-Verlag, Berlin, 1975, pp. 25–70.

[30] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin, Heidelberg, New York, 1980.

[31] S. Kawashima, *Large-time behavior of solutions for hyperbolic-parabolic systems of conservation laws*, Proc. Japan Acad. Ser. A, 62 (1986), pp. 285–287.

[32] C. A. Klausmeier, *Regular and irregular patterns in semi-arid vegetation*, Science, 284 (1999), pp. 1826–1828.

[33] M. Köhne, J. Prüss, and M. Wilke, *On quasilinear parabolic evolution equations in weighted L_p-spaces*, J. Evol. Equ., 10 (2010), pp. 443–463.

[34] N. Kumar and W. Horsthemke, *Turing bifurcation in a reaction-diffusion system with density-dependent dispersal*, Phys. A, 389 (2010), pp. 1812–1818.

[35] O. Ladyzhenskaya, V. Solonnikov, and N. Ural'tseva, *Linear and Quasi-linear Equations of Parabolic Type*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 1968.

[36] Y. Latushkin, J. Prüss, and R. Schnaubelt, *Stable and unstable manifolds for quasilinear parabolic systems with fully nonlinear boundary conditions*, J. Evol. Equ., 6 (2006), pp. 537–576.

[37] Y. Latushkin, J. Prüss, and R. Schnaubelt, *Center manifolds and dynamics near equilibria of quasilinear parabolic systems with fully nonlinear boundary conditions*, Discrete Contin. Dyn. Syst. Ser. B, 9 (2008), pp. 595–633.
[38] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications* II, Springer, Berlin, 1972.
[39] A. Lunardi, *Interpolation Theory*, Publ. Sc. Norm. Super. 9, Springer, New York, 2009.
[40] A. Lunardi, *Analytic Semigroups and Optimal Regularity for Parabolic Problems*, Progr. Nonlinear Differential Equations Appl. 16, Birkhäuser Boston, Boston, 1995.
[41] P. K. Maini, M. R. Myerscough, J. D. Murray, and K. H. Winters, *Bifurcating spatially heterogeneous solutions in a chemotaxis model for biological pattern formation*, Bull. Math. Biol., 53 (1991), pp. 701–719.
[42] M. Meyries, *Local well-posedness and instability of travelling waves in a chemotaxis model*, Adv. Differential Equations, 16 (2011), pp. 31–60.
[43] A. Mielke, *Instability and stability of rolls in the Swift–Hohenberg equation*, Comm. Math. Phys., 189 (1997), pp. 829–853.
[44] D. S. Morgan, A. Doelman, and T. J. Kaper, *Stationary periodic patterns in the* 1*D Gray-Scott model*, Methods Appl. Anal., 7 (2000), pp. 105–150.
[45] T. Naga and T. Ikeda, *Traveling waves in a chemotactic model*, J. Math. Biol., 30 (1991), pp. 169–184.
[46] R. Nagel, *Towards a "matrix theory" for unbounded operator matrices*, Math. Z., 201 (1989), pp. 57–68.
[47] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations, 55 (1984), pp. 225–256.
[48] K. J. Palmer, *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc., 104 (1988), pp. 149–156.
[49] J. Prüss, *Maximal regularity for evolution equations in L_p-spaces*, Conf. Sem. Mat. Univ. Bari, 285 (2003), pp. 1–39.
[50] J. Prüss, G. Simonett, and R. Zacher, *On convergence of solutions to equilibria for quasilinear parabolic problems*, J. Differential Equations, 246 (2009), pp. 3902–3931.
[51] J. Prüss, G. Simonett, and R. Zacher, *On normal stability for nonlinear parabolic problems*, Discrete Contin. Dynam. Systems, Supplement (2009), pp. 612–621.
[52] F. Räbiger and R. Schnaubelt, *The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions*, Semigroup Forum, 52 (1996), pp. 225–239.
[53] J. D. M. Rademacher, B. Sandstede, and A. Scheel, *Computing absolute and essential spectra using continuation*, Phys. D, 229 (2007), pp. 166–183.
[54] M. R. Roussel and J. Wang, *Transition from self-replicating behavior to stationary patterns induced by concentration-dependent diffusivities*, Phys. Rev. Lett., 87 (2001), pp. 188–302.
[55] C. J. Roussel and M. R. Roussel, *Reaction–diffusion models of development with state-dependent chemical diffusion coefficients*, Progr. Biophys. Molec. Biol., 86 (2004), pp. 113–160.
[56] B. Sandstede and A. Scheel, *Relative Morse indices, Fredholm indices, and group velocities*, Discrete Contin. Dyn. Syst. Ser. A, 20 (2008), pp. 139–158.
[57] B. Sandstede and A. Scheel, *On the structure of spectra of modulated travelling waves*, Math. Nachr., 232 (2001), pp. 39–93.
[58] B. Sandstede and A. Scheel, *Defects in oscillatory media: Toward a classification*, SIAM J. Appl. Dyn. Syst., 3 (2004), pp. 1–68.
[59] B. Sandstede, *Stability of travelling waves*, in Handbook of Dynamical Systems, II, B. Fiedler, ed., Elsevier, New York, 2002, pp. 983–1055.
[60] G. Schneider, *Nonlinear diffusive stability of spatially periodic solutions—Abstract theorem and higher space dimensions*, in Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), Tohoku Mathematical Publishing, Sendai, Japan, 1998, Vol. 8, pp. 159–167.
[61] J. Shatah and W. Strauss, *Spectral condition for instability*, Contemp. Math., 255 (2000), pp. 189–198.
[62] E. Sheffer, H. Yizhaq, M. Shachak, and E. Meron, *Mechanisms of vegetation-ring formation in water-limited systems*, J. Theoret. Biol., 273 (2011), pp. 138–146.
[63] L. de Simon, *Un applicazione del la teoria degli integrali singolari al lo studio del le equazioni differenziali lineari astratte del primo ordine*, Rend. Sem. Mat. Univ. Padova, 34 (1964), pp. 205–223.

[64] G. Simonett, *Center manifolds for quasilinear reaction-diffusion systems*, Differential Integral Equations, 8 (1995), pp. 753–796.

[65] S. van der Stelt, A. Doelman, G. Hek, and J. D. M. Rademacher, *Rise and fall of periodic patterns for a generalized Klausmeier-Gray-Scott model*, J. Nonlinear Sci., 23 (2013), pp. 39–95.

[66] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North–Holland, Amsterdam, 1978.

[67] H. Triebel, *Theory of Function Spaces*, II, Birkhäuser, Basel, 1983.

[68] H. Uecker, *Self-similar decay of spatially localized perturbations of the Nusselt solution for the inclined film problem*, Arch. Ration. Mech. Anal., 184 (2007), pp. 401–447.

[69] A. Yagi, *Abstract Parabolic Evolution Equations and Their Applications*, Springer, Berlin, Heidelberg, 2010.

[70] Y. Wu and X. Zhao, *The existence and stability of travelling waves with transition layers for some singular cross-diffusion systems*, Phys. D, 200 (2005), pp. 325–358.

[71] K. Zumbrun, *Center Stable Manifolds for Quasilinear Parabolic PDE and Conditional Stability of Nonclassical Viscous Shock Waves*, preprint, Department of Mathematics, Indiana University, 2008; available as arXiv:0811.2788.

[72] K. Zumbrun, *Planar stability criteria for viscous shock waves of systems with real viscosity*, in Hyperbolic Systems of Balance Laws, Lecture Notes in Math. 1911, Springer, Berlin, 2007, pp. 229–326.

[73] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana Univ. Math. J., 47 (1998), pp. 741–872.