Instabilities of Wave Trains and Turing Patterns in Large Domains

Jens D.M. Rademacher
Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39
10117 Berlin, Germany

Arnd Scheel
University of Minnesota
School of Mathematics
206 Church St. S.E.
Minneapolis, MN 55455, USA

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Dedicated to André Vanderbauwhede on the occasion of his 60th birthday.

Abstract

We classify generic instabilities of wave trains in reaction-diffusion systems on the real line as the wavenumber and system parameters are varied. We find three types of robust instabilities: Hopf with nonzero modulational wavenumber, sideband, and spatio-temporal period-doubling. Near a fold, the only other robust instability mechanism, we show that all wave trains are necessarily unstable. We also discuss the special cases of homogeneous oscillations and reflection symmetric, stationary Turing patterns.

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Corresponding author: Jens Rademacher rademach@wias-berlin.de

1 Wave Trains in Reaction-Diffusion Systems

We are interested in spatio-temporally periodic solutions in essentially one-dimensional systems of partial differential equations on large domains. As a prototype, we study reaction-diffusion systems on the real line

$$u_t = Du_{xx} + f(u; \mu), \tag{1}$$

where $u \in \mathbb{R}^N$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, $D = \mathrm{diag}\,(d_j) \geq 0$, and the nonlinearity f, referred to as the kinetics, has sufficiently many derivatives. Our approach translates in a straightforward manner to more complicated, dissipative physical systems such as damped-driven fluid motion or pattern formation in optics.

The simplest patterns beyond spatially homogeneous equilibria are solutions which break one of the two continuous symmetries, time and space translation, with a residual discrete spatio-temporal symmetry, $u = u_{\mathrm{wt}}(kx - \omega t)$, $u(\xi) = u(\xi + 2\pi)$. We refer to this type of solutions, which are stationary in the comoving variable $y = x - \frac{\omega}{k}t$, as *wave trains*. From a symmetry point of view, wave trains possess maximal nontrivial isotropy in the group $\mathbb{R} \times \mathbb{R}$ of space and time translations.

There are two special cases, $k = 0$ and $\omega = 0$: the case $k = 0$ corresponds to a periodic solution of the pure kinetics ODE, in absence of diffusion, which are spatially *homogeneous oscillations*. The case $\omega = 0$ corresponds to spatially periodic stationary solutions, which we refer to as *Turing patterns* [Turing, 1952]. These special cases are also distinguished in terms of symmetry, because (1) possesses a reflection symmetry $x \to -x$ which leaves homogeneous oscillations and Turing patterns invariant. In fact, here and in the following we will *assume* that Turing patterns are reflection symmetric. This is typically the case since non-symmetric wave trains with $\omega = 0$ are not robust under changes of wavenumber or parameter. Indeed, nonreversible periodic orbits of the associated ODE are typically isolated, hence possess a selected wavenumber. Orbits with nearby wavenumber require adjusting $\omega \neq 0$. Reversible periodic orbits come in families, typically parameterized by the period, so $\Omega(k) \equiv 0$.

While wave trains are an interesting and common phenomenon in pattern-forming reaction-diffusion systems such as variants of the Belousov-Zhabotinsky [De Wit, 1999] reaction or the Chlorite-Iodite-Malonic-Acid reaction [Castets et al., 1990], they are predominant in a number of other non-equilibrium, damped-driven physical systems. Here we only mention waves in various fluid experiments such as Marangoni convection, Bénard convection, and the Taylor-Couette experiments;
[Cross & Hohenberg, 1993, Bodenschatz et al., 2000, Chossat & Iooss, 1994]. In order to map out qualitatively different dynamics in these extended systems, the stability of wave trains and how stability is typically lost are of particular interest. In this article we give a list of codimension-one instabilities of wave trains in terms of the modulational frequency and wavenumber in the critical spectrum, and determine which of these instabilities may occur in a robust fashion for wave trains, homogeneous oscillations, and Turing patterns.

Typical instabilities are:

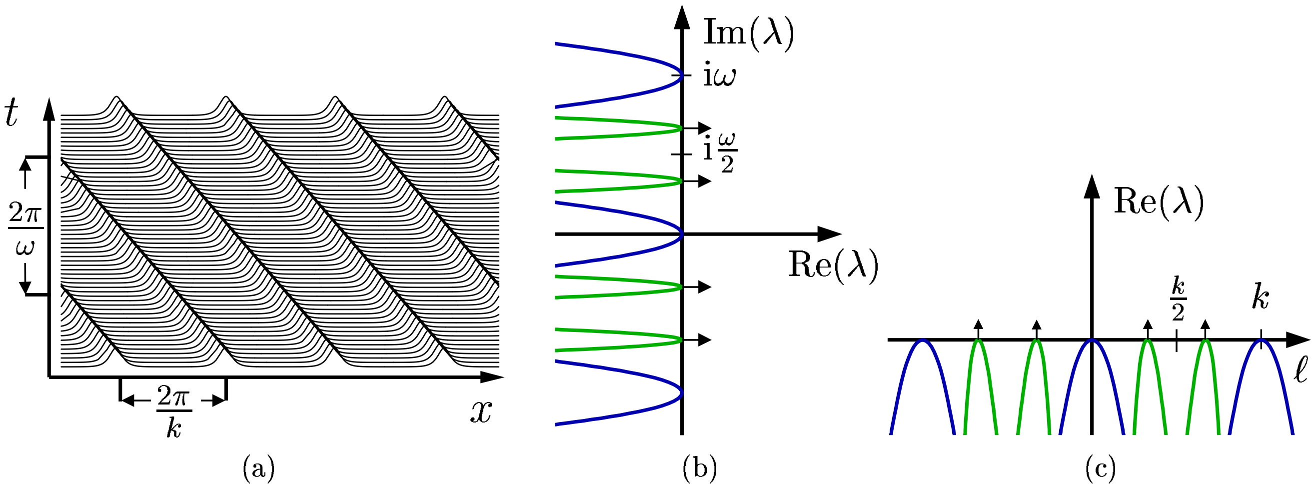

Figure 1: Illustration of the Hopf case of a generic wave train with frequency ω and wavenumber k. (a) sketch of space-time plot, (b) critical spectrum $\lambda \in \mathbb{C}$ in the complex plane, (c) parametrization of the curves $\lambda(\ell)$ by ℓ. Arrows in (b) and (c) indicate the motion as μ moves through the instability.

Hopf:	Critical nonzero frequency, and non-zero, non-resonant wave number.
Period Doubling:	Hopf with 1:2 resonance in wave number *and* frequency.
Sideband:	Critical frequency and wave number close to zero.
Pure Hopf:	Hopf with zero wave number.
Fold or Pitchfork:	Critical frequency zero in addition to translational mode.
Turing:	Critical frequency zero and wave number non-zero.

Only three of these mechanisms are typical for generic wave trains, while all of them are robust in reflection symmetric scenarios; see Table 1. For illustration we sketch the space-time profiles and critical spectrum at a Hopf bifurcation of a generic wave train in Fig. 1.

Our main results give proofs of robustness. We also argue for non-robustness, but only on the level of the dispersion relation. The dispersion relation is an analytic function of frequency and wavenumber, whose roots correspond to spectrum of the linearization at a wave train of (1). Our results extend the simpler list for the onset of instability at spatially homogeneous equilibrium points, $k = \omega = 0$, where unstable modes come as

	Hopf $\alpha \neq 0$ $\ell \neq 0$	Per. dbl. $\alpha = \omega/2$ $\ell = k/2$	Sideband $\alpha = 0$ $\ell = 0$	Pure Hopf $\alpha \neq 0$ $\ell = 0$	Fold $\alpha = 0$ $\ell = 0$	Turing $\alpha = 0$ $\ell \neq 0$
Generic $\omega, k \neq 0$	$\surd$	$\surd$	$\surd$			
Oscillation $\omega \neq 0, k = 0$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Turing $\omega = 0, k \neq 0$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Equilibria $\omega = k = 0$	$\surd$			$\surd$	$\surd$	$\surd$

Table 1: List of robust codimension-one instabilities with respect to a mode $\mathrm{e}^{\mathrm{i}(\alpha t - \ell x)}$ of a wave train with frequency ω and wavenumber k.

$\mathrm{e}^{\mathrm{i}(\alpha t - \ell x)}$, leading to the *four elementary instabilities* Fold, Turing, Pure Hopf, and oscillatory Turing (sometimes referred to as Turing-Hopf, here simply 'Hopf'); [Scheel, 2003, Def. 2.3].

Outline: We start with a review of robustness properties of families of wave trains in Sec. 2. We then characterize linear stability and classify boundaries of stability in Sec. 3. We discuss different directions in which this linear stability analysis can be complemented to a nonlinear bifurcation theory, including modulation equations, bifurcation theory in spatial and temporal dynamics, and absolute instabilities in Sec. 4.

2 Families of Wave Trains

Wave trains solve the boundary-value problem

$$k^2 D u'' + \omega u' + f(u; \mu) = 0\,, \quad u(2\pi) = u(0), \tag{2}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}x}$, and the linearization in a wave train u_{wt}, given by

$$\mathcal{L}u := k^2 D u'' + \omega u' + \partial_u f(u_{\mathrm{wt}}; \mu) u, \tag{3}$$

defines a closed unbounded operator with compact resolvent on $L^2_{\mathrm{per}}(0, 2\pi)$ with domain $H^2_{\mathrm{per}}(0, 2\pi)$ when $k \neq 0$ and $H^1_{\mathrm{per}}(0, 2\pi)$ when $k = 0$ and $\omega \neq 0$. Note that u'_{wt} always contributes to the kernel of $\mathcal{L}$ owing to the spatial translation symmetry of (2).

Lemma 2.1 (Continuation of wave trains) *Assume that (2) has a wave train solution u^*_{wt} for some $(k, \omega, \mu) = (k_*, \omega_*, \mu_*)$ where either $k_* \neq 0$ or $\omega_* \neq 0$. Suppose* *that $\lambda = 0$ is an eigenvalue of $\mathcal{L}$ of algebraic multiplicity 1. Then there exists a local family of wave trains $u_{\rm wt}(\xi; k, \mu)$ with frequencies $\omega(k, \mu)$ for $(k, \mu) \sim (k_*, \mu_*)$, which are continuously differentiable in (k, μ) with $u_{\rm wt}(\xi; k_*, \mu_*) = u^*_{\rm wt}(\xi)$ and $\omega(k_*, \mu_*) = \omega_*$.*

Proof. In case $k_* \neq 0$, the derivative of (2) with respect to ω gives u', an element of the kernel of $\mathcal{L}$, and by the assumption of algebraic multiplicity one of 0 in Ker $(\mathcal{L})$, it does not lie in its range. By the implicit function theorem we can solve for ω and u jointly as a function of parameters such as k and μ.

In case $k_* = 0$ we have $\omega_* \neq 0$ by assumption, and so we may precondition (2) for $(k, \omega) \sim (k_*, \omega_*)$ by writing

$$\partial_\xi u + (k^2 D \partial_\xi + \omega)^{-1} f(u; \mu) = 0.$$

Since $f : H^1_{\rm per}(0, 2\pi) \to H^1_{\rm per}(0, 2\pi)$ is smooth and $(k^2 D \partial_\xi + \omega)^{-1} : H^1_{\rm per}(0, 2\pi) \to L^2_{\rm per}(0, 2\pi)$ is continuously differentiable[1] in $k = 0$ for $\omega \neq 0$ (a direct consequence of the Fourier space representation), we can again solve via the implicit function theorem by exploiting simplicity of the zero eigenvalue of the linearization at $k = 0$. ■

Remark 2.2 *If $u^*_{\rm wt}$ is reflection symmetric, $u^*_{\rm wt}(\xi) = u^*_{\rm wt}(-\xi)$, $k \neq 0$, $\omega = 0$, then (2) can be solved in the space of even functions for a family of even patterns. The spectral assumption can thus be reduced to an assumption in the subspace of even functions.*

We call the function $\omega = \Omega(k)$ the *nonlinear dispersion relation*, the quotient $\frac{\omega}{k} =: c_{\rm ph}$ the *phase speed* and $\Omega'(k)$ the (nonlinear) *group velocity.*

From a slightly different view point we rewrite the boundary-value problem (2) as a first-order ODE

$$u' = v, \qquad k^2 v' = -D^{-1} \left(\omega v - f(u; \mu) \right). \tag{4}$$

and look for periodic orbits to this system with parameter ω given as a function of the spatial period $\frac{2\pi}{k}$. Due to the spatial translation symmetry of (2), the derivative (u', v') gives rise to a trivial Floquet multiplier $\rho = 1$ near the periodic orbit. The geometric multiplicity of ρ coincides with the geometric multiplicity of $\lambda = 0$ for $\mathcal{L}$, however algebraic multiplicity one of the unit Floquet multiplier ρ amounts to a different condition than algebraic multiplicity one of $\lambda = 0$ for $\mathcal{L}$: assuming ρ is algebraically simple, we would be able to solve for k (alias the period) as a function of ω and μ, which appear explicitly in (4). Also, the limit $k = 0$ is somewhat more difficult as it gives rise to a singular perturbation problem with slow manifold $v = f(u; \mu)/\omega$ and slow flow $u' = f(u; \mu) + \mathrm{O}(k^2)$.

[1] A bootstrap argument implies more smoothness, but we do not need it for our purposes.

On the other hand, the symmetry $x \mapsto -x$ translates into a *reversibility* [Ciocci & Vandebauwhede, 2004] for (4) at $\omega = 0$: solutions $(u,v)(\xi)$ yield solutions $(u,-v)(-\xi) = R(u,v)(-\xi)$, with involution $R : (u,v) \mapsto (u,-v)$ and symmetry plane $\mathrm{Fix}\, R = \{(a,0)\,|\, a \in \mathbb{R}\}$. Reversible periodic orbits, that is, periodic orbits that are invariant under R as a set, come in one-parameter families if the images of $\mathrm{Fix}\, R$ under the period maps $\Phi_{2\pi/k}$, $k \sim k_*$ of the flow Φ_x of (4) intersect $\mathrm{Fix}\, R$ transversely. We may then vary ω or μ and find periodic orbits nearby.

We emphasize that the class of spatio-temporally periodic solutions of a single phase variable does not include the interesting class of *standing waves*, $u(t,x) = u(t+T,x) = u(t,-x) = u(t,x+L)$, which are even and doubly periodic in x and t.

3 Stability: ODE and PDE Spectra

The stability of a given wave train in the PDE is determined largely by spectral information. We therefore study the eigenvalue problem to the linearization of (1) at a wave train $u_{\mathrm{wt}}(kx-\omega t)$,

$$u_t = Du_{xx} + f'(u_{\mathrm{wt}}(kx-\omega t;\mu)u. \tag{5}$$

When $\omega \neq 0$, this parabolic equation possesses a period-$\frac{2\pi}{\omega}$ map Φ and we denote the spectrum of Φ considered on $L^2(\mathbb{R})$ by Σ_{m}. For convenience, we will mostly work with Floquet exponents, i.e. the set $\Sigma = \{\lambda;\ \mathrm{e}^{2\pi\lambda/\omega} \in \Sigma_{\mathrm{m}}\}$. In order to describe the set Σ, it suffices to consider solutions of (5) of the form

$$u(t,x) = \mathrm{e}^{\lambda t + \nu x} w(kx-\omega t), \qquad w(\xi) = w(\xi + 2\pi).$$

so that w satisfies

$$\mathcal{L}(\nu)w := D(k\partial_\xi + \nu)^2 w + \omega\partial_\xi w + f'(u_{\mathrm{wt}}(\xi);\mu)w = \lambda w. \tag{6}$$

We let $\Sigma(\nu)$ denote the spectrum of $\mathcal{L}(\nu)$ as an unbounded operator on $L^2_{\mathrm{per}}(0,2\pi)$.

In case $\omega = 0$, equation (5) is autonomous and the spectrum of an arbitrary time-T map is given by the exponential of the spectrum Σ of the operator on the right-hand side of (5).

Lemma 3.1 (Floquet-Bloch decomposition) *It holds that*

$$\Sigma = \bigcup_{\nu \in \mathrm{i}\mathbb{R}} \Sigma(\nu) = \bigcup_{\nu \in \mathrm{i}[0,k)} \Sigma(\nu).$$

Proof. A proof in the steady case $\omega = 0$ is contained in [Gardner, 1993] and the relation to the period map in the time-periodic case in [Sandstede & Scheel, 2001a]. ∎

The lemma allows to define geometric and algebraic multiplicities for an element λ in the spectrum Σ as the sum of all geometric or algebraic multiplicities of λ in $\Sigma(\nu)$ for all $\nu \in \mathrm{i}[0, k)$ so that $\lambda \in \Sigma(\nu)$.

We can obtain a very compact characterization of the eigenvalue problem by writing (6) as a first-order ODE

$$k\partial_\xi w = -\nu w + v, \qquad k\partial_\xi v = -\nu v + D^{-1}\left(-\omega\frac{\nu}{k}w + \frac{\omega}{k}v - f'(u_{\mathrm{wt}}(\xi)w + \lambda w\right).$$

We denote by $\Psi_{\lambda,\nu}$ the period-2π map of the non-autonomous ODE. The spectrum $\Sigma(\nu)$ consists precisely of those λ for which $\Psi_{\lambda,\nu}$ possesses a nontrivial fixed point.

Lemma 3.2 (Complex dispersion relation) *The spectrum of $\Sigma(\nu)$ is given by the roots λ of*

$$d(\lambda, \nu) := \det(\Psi_{\lambda,\nu}),$$

and the algebraic multiplicity of the eigenvalue coincides with the order of the root of d.

Proof. The spectral characterization by roots follows also from the Floquet-Bloch decomposition, and multiplicities coincide due to Jordan chains of the same length as the order of the roots [Gardner, 1993]. ∎

For $\lambda = \mathrm{i}\alpha$ and $\nu = \mathrm{i}\ell$ we obtain the usual linear dispersion relation $d(\mathrm{i}\alpha, \mathrm{i}\ell) = 0$, which we can typically solve locally for $\alpha(\ell)$; we call $c_{\mathrm{g}} := \frac{\mathrm{d}}{\mathrm{d}\ell}|_{\ell=0}\alpha(\ell)$ the (linear) *group velocity*, which coincides with the nonlinear group velocity $\Omega'(k) = c_{\mathrm{g}}$; see [Doelman et al., 2006]. The linear group velocity measures the speed of transport of a *small* perturbation, while the nonlinear group velocity measures the change in the speed of propagation when perturbing to an *exact* nearby wave train — which is *not* a small perturbation globally. However, a simple perturbation analysis immediately shows that both quantities coincide. Since the boundary-value problem (6) is real and inherits possible symmetries of $u_{\mathrm{wt}}(\xi)$, we have

$$\overline{d(\lambda, \nu)} = d(\bar{\lambda}, \bar{\nu}), \tag{7}$$

and, if $\omega = 0$ and u is even, or if $k = 0$, we have

$$d(\lambda, \nu) = d(\lambda, -\nu). \tag{8}$$

The Floquet covering symmetry induced by the logarithm translates into

$$d(\lambda,\nu)=d(\lambda-\mathrm{i}\omega,\nu+\mathrm{i}k), \tag{9}$$

because for a solution $w(\xi)=\mathrm{e}^{\mathrm{i}\xi}\tilde{w}(\xi)$ of (6) we have $\mathcal{L}(\nu)w=\lambda w$ as well as

$$\mathcal{L}(\nu)\mathrm{e}^{\mathrm{i}\xi}\tilde{w}(\xi)=\mathcal{L}(\nu+\mathrm{i}k)w+\mathrm{i}\omega w,$$

and hence $\mathcal{L}(\nu+\mathrm{i}k)w=(\lambda-\mathrm{i}\omega)w$.

Instabilities of spectrum through the imaginary axis are characterized by purely imaginary $\lambda=\mathrm{i}\alpha$ and $\nu=\mathrm{i}\ell$, which thus come as a sequence $(\pm\mathrm{i}\alpha+\mathrm{i}m\omega,\pm\mathrm{i}\ell-\mathrm{i}mk)$, $m\in\mathbb{Z}$. We remark that in the comoving spatial coordinate $y=x-\frac{\omega}{k}t$ the covering symmetry of the spectrum becomes $(\lambda,\nu+\mathrm{i}m\frac{\omega}{k})$, $m\in\mathbb{Z}$, so marginally stable spectrum typically consists of a single point or complex conjugate pair. In this setting closed curves of spectrum are possible, but in coordinates $y=x-ct$ with $c\neq\frac{\omega}{k}$ these become unbounded spiraling curves in the complex plane with periodic real part.

Since $d(0,0)=0$, the following definition characterizes the "most stable", and robust spectral configuration for a wave train.

Definition 3.3 *A wave train u_{wt} is called* strongly stable*, if its dispersion relation satisfies the following conditions.*

(S) ***Stable for $\lambda\neq 0$:*** $d(\lambda,\mathrm{i}\ell)\neq 0$ *for all* $\mathrm{Re}\,\lambda\geq 0$, $\lambda\neq 0$, *and all* $\ell\in\mathbb{R}$

(S0) ***Stable for $\ell\neq 0$:*** $d(0,\mathrm{i}\ell)\neq 0$ *for all* $\ell\in(0,k)$

(LS) ***Long-wave stability:*** $\partial_\lambda d(0,0)\partial_{\nu\nu}d(0,0)<0$

Note that (LS) corresponds to stability against long wave length perturbations. According to the literature, we refer to failure of (LS) as a sideband instability, since a band of spectrum $\ell\sim 0$ *neighboring* the neutral mode $\ell=0$ generated by the wave train itself generates the instability; see for instance [Cross & Hohenberg, 1993].

Our interest is the boundary of the set of strongly stable wave trains; we will argue in terms of codimension and classify the codimension-one scenarios which might be encountered during a parameter homotopy, fixing either k or μ.

More precisely, we say that an instability is *robust*, if the conditions for its onset are codimension-1, i.e. in an m-parameter family of dispersion relations with the above properties the conditions are satisfied on a $m-1$-dimensional manifold under generic conditions on the unfolding. Note that it is sufficient to show robustness under perturbations of d, since d depends smoothly on the coefficients of the ODE.

Notation: Whenever $\partial_\lambda d \neq 0$ for some $\lambda = \mathrm{i}\alpha$ and $\nu = \mathrm{i}\ell$, we can solve $d(\lambda,\nu) = 0$ for $\lambda = \lambda(\nu)$ locally. We will refer to coefficients in the Taylor jet of this complex curve $\lambda(\nu)$ via

$$\lambda(\nu - \mathrm{i}\ell) = \mathrm{i}a_0 + a_1(\nu - \mathrm{i}\ell) + a_2(\nu - \mathrm{i}\ell)^2 + a_3(\nu - \mathrm{i}\ell)^3 + a_4(\nu - \mathrm{i}\ell)^4 + R(\nu - \mathrm{i}\ell), \quad (10)$$

where $R(\nu) = \mathrm{O}((\nu - \mathrm{i}\ell)^5)$, and $a_0 = \alpha \in \mathbb{R}$. The coefficients $a_j \in \mathbb{C}$ can be readily computed by implicit differentiation, and inherit parameter dependence. Note that whenever in addition $\lambda(0) = 0$, complex conjugation symmetry implies $a_j \in \mathbb{R}$.

3.1 Generic wave trains, $c_\mathrm{g} \neq 0$, $k \neq 0$

The boundary of strong stability is characterized by four different conditions: $d(\mathrm{i}\alpha, \mathrm{i}\ell) = 0$ for $\alpha = 0$, $\alpha \neq 0$, or $\partial_\lambda d(0,0) = 0$, or $\partial_{\nu\nu} d(0,0) = 0$. We claim that the following three conditions are robust in the boundary of strong stability for wave trains with $c_\mathrm{g} \neq 0$ and $k \neq 0$, that is, they occur in open subsets of the boundary of stability.

Hopf: We assume (LS) and (S0), and we assume that (S) is violated for a unique pair (up to Floquet multiples) $\lambda = \pm\mathrm{i}\alpha \notin \{0, \mathrm{i}\frac{\omega}{2}\}$, $\nu = \pm\mathrm{i}\ell \notin \{0, \mathrm{i}\frac{k}{2}\}$, where $d = 0$ and for (10) we have

$$a_1 \in \mathbb{R} \setminus \{0\}, \quad \mathrm{Re}(a_2) > 0.$$

Period-Doubling: We assume (LS) and (S0), and that (S) is violated for a unique (up to Floquet multiples) $\lambda = \mathrm{i}\frac{\omega}{2}$, $\nu = -\mathrm{i}\frac{k}{2}$, where $d = 0$ and for (10) we have

$$a_1 \in \mathbb{R} \setminus \{0\}, \quad \mathrm{Re}(a_2) > 0.$$

Sideband: We assume (S) and (S0), and that (LS) is violated so that at $\lambda = \nu = 0$ for (10) we have

$$a_2 = 0, \quad a_4 < 0.$$

The following lemma gives assumptions on the unfolding that give a robust instability. We denote by μ the unfolding parameter, and by $\hat{\mu}$ possible perturbation parameters.

Lemma 3.4 *Hopf and Period-Doubling are robust if* $\mathrm{Re}\,\partial_\mu(a_0) \neq 0$*, and Sideband if* $\partial_\mu(a_2) \neq 0$.

Proof. We can always write the perturbed curve of critical eigenvalues as $\lambda(\nu) = a_0(\mu) + a_1(\mu)\hat{\nu} + a_2(\mu)\hat{\nu}^2 + \mathrm{O}(3)$, where $\hat{\nu} = \nu - \mathrm{i}\ell$.

In the Hopf case, $\alpha \neq 0$, $a_0(0) = \mathrm{i}\alpha$. The onset of stability is depending on μ

implicitly through the equations

$$\operatorname{Re}\nu = 0, \quad \operatorname{Re}\lambda(\nu) = 0, \quad \operatorname{Im}\lambda'(\nu) = 0. \tag{11}$$

We can solve these three (real) equations with respect to the (real) parameters $\mu, \operatorname{Re}\nu, \operatorname{Im}\nu$. Indeed, we can solve the first two equations by adjusting $\operatorname{Re}\nu$ (trivially) and μ (by assumption). We are then left with $\operatorname{Im}(a_1 + a_2\nu + \mathrm{O}(\nu^2)) = 0$, evaluated in $\nu = \mathrm{i}\ell$. Since $\operatorname{Re} a_2(0) \neq 0$, we may solve for $\operatorname{Im}\nu$ as a function of μ by the implicit function theorem. In particular, the solutions to the set of equations (11) depend smoothly on additional perturbation parameters $\hat{\mu}$, which shows robustness. Note that by continuity, the stability condition $\operatorname{Re}(a_2) > 0$ is met for small $\hat{\mu}$ as well.

In the period-doubling case, application of complex conjugation (7) and Floquet symmetry (9) shows that

$$\begin{aligned} d_s(\lambda, \nu) &:= d\left(\frac{\mathrm{i}\omega}{2} + \lambda, \nu - \frac{\mathrm{i}k}{2}\right) = \overline{d\left(-\frac{\mathrm{i}\omega}{2} + \bar{\lambda}, \frac{\mathrm{i}k}{2} + \bar{\nu}\right)} \\ &= \overline{d\left(\frac{\mathrm{i}\omega}{2} + \bar{\lambda}, -\frac{\mathrm{i}k}{2} + \bar{\nu}\right)} = \overline{d_s(\bar{\lambda}, \bar{\nu})}, \end{aligned}$$

so that d_s maps $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. The conditions $d_s(0, 0; \mu) = 0$,

$$\partial_\mu d_s(0, 0; \mu) = \partial_\mu d\left(\mathrm{i}\frac{\alpha}{2}, -\mathrm{i}\frac{\ell}{2}; \mu\right) \neq 0,$$

give a robust zero of the real function $d_s(\lambda, 0; 0) = 0$; note that $(\omega, k) = (\omega(\mu), k(\mu))$. Hence, perturbations of d cannot change the imaginary part of the perturbed roots, and the same condition as for Hopf implies robustness with the additional constraint.

In the sideband case, note that all coefficients in (10) are real at $\nu = 0$, and that $\lambda(0; \cdot) \equiv 0$ is enforced by translation symmetry. Hence, it is sufficient to adjust the quadratic coefficient, which is guaranteed by our assumption. ■

Sideband instabilities and Hopf bifurcation are known to occur in the complex Ginzburg-Landau equation, where wave trains are actually relative equilibria with respect to the gauge symmetry: spatio-temporal profiles of wave trains are obtained by the gauge symmetry, complex rotation in the dependent variable only. Period-Doubling occurs for wave trains with small wavenumber $k \sim 0$ whenever the homogeneous oscillations undergo a period-doubling bifurcation in the kinetics and has been observed experimentally in the BZ-reaction [Yoneyama et al., 1995].

Also, each of these bifurcations occurs in long wave-length limits, where the wave trains converge to pulse trains. In the traveling wave equation in the comoving frame

$y = x - \frac{\omega}{k}t$ periodic orbits converge to a homoclinic orbit while $\frac{\omega}{k}$ converges to the speed of the pulse in this limit. If the pulse possesses a weakly decaying oscillatory tail, the associated homoclinic orbit is of Shil'nikov-type and spatial dynamics are particularly rich, including in particular period-doubling cascades [Shil'nikov et al., 1998, Shil'nikov et al, 2001]. Following these periodic orbits, one can study the stability as solutions of the PDE [Sandstede & Scheel, 2001b] and find period-doubling as well as side-band instabilities. An even richer scenario is encountered in the presence of reflection symmetry, where the traveling wave equation is reversible [Vanderbauwhede & Fiedler, 1992, Fiedler & Turaev, 1996].

Without striving for the most general genericity result, we now give some results that indicate why we believe our list contains all robust instabilities. The following lemma shows in particular that, on the level of the dispersion relation d, the cases $\ell = 0$, *Pure Hopf*, or $\alpha = 0$, *Turing*, are not robust.

Lemma 3.5 *Let $d(\lambda, \nu; \mu)$ be a dispersion relation satisfying complex conjugation and Floquet-covering symmetries, with a stable dispersion curve at the origin, i.e. (LS) holds, and undergoing a Hopf instability at $\lambda = \mathrm{i}\alpha$, $\nu = \mathrm{i}\ell$, with $(\alpha, \ell) \notin \{(0,0), (\frac{\omega}{2}, \frac{k}{2})\}$. Then we can find arbitrarily close dispersion relations $\tilde{d}$ with the same properties, such that at onset $\alpha \neq \tilde{\alpha}$, and $\ell \neq \tilde{\ell}$.*

Proof.

Choose $\varepsilon > 0$, small, and consider the modified dispersion relation $d + \varepsilon \sum_j d_j$. We will make a sequence of choices for d_j, all of which satisfy the complex conjugation and Floquet covering symmetry, and respect the translation zero in $\lambda = \nu = 0$. Moreover, the perturbations d_j will be bounded as $\operatorname{Re}\lambda \to +\infty$ and therefore do not create new unstable eigenvalues for small ε.

We first choose

$$d_1(\lambda) = c_1(\mathrm{e}^{-2\pi\lambda/\omega} - 1) + c_2(\mathrm{e}^{-2m\pi\lambda/\omega} - 1),$$

with arbitrary constants $c_j \in \mathbb{R}$ and $m \in \mathbb{Z}$. Near $\lambda = \mathrm{i}\alpha$, we have

$$d_1 = c_1(\mathrm{e}^{-2\pi\mathrm{i}\alpha/\omega} - 1) + c_2(\mathrm{e}^{-2m\pi\mathrm{i}\alpha/\omega} - 1) + \mathrm{O}(\lambda - \mathrm{i}\alpha).$$

For $\alpha \notin \{0, \frac{\omega}{2}\}$ we can find $m \in \mathbb{Z}$ so that with appropriate choice of c_j, $j = 1, 2$, we can assign to $d_1(\mathrm{i}\alpha)$ an arbitrary complex number.

Next, consider the perturbation

$$d_2(\nu) = c_1(\mathrm{e}^{2\pi\nu/k} - 1) + c_2(\mathrm{e}^{2m\pi\nu/k} - 1),$$

Again, we can assign an arbitrary complex number to $d_2(\mathrm{i}\ell)$, provided $\ell \notin \{0, \frac{k}{2}\}$.

We now consider

$$d_3(\lambda, \nu) = c_1(\mathrm{e}^{-2c_2(\lambda/\omega+\nu/k)} - 1).$$

For $(\lambda, \nu) = (\mathrm{i}\frac{\omega}{2}, 0)$ and $(\lambda, \nu) = (0, \mathrm{i}\frac{k}{2})$, d_3 evaluates to $c_1\mathrm{e}^{-\mathrm{i}c_2}$, again an arbitrary complex number. Summarizing, we have shown that in all cases $(\alpha, \ell) \notin \{(0,0), (\frac{\omega}{2}, \frac{k}{2})\}$ we can add an arbitrary value to d at the critical root with a suitable perturbation.

Given the local expansion of d at $\hat{\lambda} - \lambda - \mathrm{i}\alpha$, $\hat{\nu} = \nu - \mathrm{i}\ell$,

$$d(\hat{\lambda}, \hat{\nu}) = d_\lambda \hat{\lambda} + d_\nu \hat{\nu} + \ldots,$$

we see that the solution curve $\hat{\lambda}(\hat{\nu})$ is changed into (with ε as above)

$$\hat{\lambda}(\hat{\nu}) = \varepsilon A + a_1\hat{\nu} + a_2\hat{\nu}^2 + \ldots, \tag{12}$$

with A arbitrary. Choosing A appropriately, we can therefore adjust the value of the critical frequency α in an arbitrary fashion with the perturbations d_1, d_2, d_3.

It remains to show that we may change the critical wavenumber $\mathrm{i}\ell$ in an arbitrary fashion. We may restrict to values of $\alpha \notin \{0, \frac{\omega}{2}\}$ by first changing the onset as described above. It is now sufficient to change the argument $\partial_\lambda d$ while fixing the root $(\mathrm{i}\alpha, \mathrm{i}\ell)$, since this would change the coefficient a_1 in the local expansion to purely imaginary, thus forcing a local maximum of $\operatorname{Re}\lambda$ in $\hat{\ell} \neq 0$. A perturbation with d_1 and A real can then be used to shift the most unstable value λ back on the imaginary axis. Consider therefore the perturbation

$$d_4^m(\lambda) = (\mathrm{e}^{-2\pi\lambda/\omega} - 1)(\mathrm{e}^{-2\pi m(\lambda-\mathrm{i}\alpha)/\omega} - 1)(\mathrm{e}^{-2\pi m(\lambda+\mathrm{i}\alpha)/\omega} - 1), \quad m = 1, 2$$

which again satisfies the symmetry and decay requirements and respects the translational mode at the origin. In addition, d_4^m respects the root $\lambda = \mathrm{i}\alpha$. Computing the derivative in the root gives

$$\partial_\lambda d_4^m = -\frac{2\pi m}{\omega}(\rho - 1)(\rho^{2m} - 1), \quad \rho = \mathrm{e}^{-2\pi\mathrm{i}\alpha/\omega} \notin \mathbb{R}.$$

Since $\rho - 1 \notin \mathbb{R}$, if $4\alpha = \omega$, and $\partial_\lambda d_4^2 / \partial_\lambda d_4^1 = 2(\rho^2 + 1)$ we have that

$$\arg \partial_\lambda d_4^1 \neq \arg \partial_\lambda d_4^2,$$

and at least one is not a multiple of π. Therefore, either d_4^1 or d_4^2 changes the argument of $\partial_\lambda d$. This completes the proof. ■

The *Fold*, where (S) and (S0) hold, but (LS) is violated with $\partial_\lambda d(0,0) = 0$, is atypical for a more subtle reason: at the fold, the dispersion relation $d(\lambda,\nu)$ possesses the expansion

$$d(\lambda,\nu) = a\lambda^2 + b\nu + \mathrm{O}(\nu^2, \lambda\nu, |\lambda|^3 + |\nu|^3),$$

with $a, b \in \mathbb{R}$. Typically, $b \neq 0$, so that for $\nu = \mathrm{i}\ell$, $\lambda = \pm c\sqrt{\mathrm{i}\ell}$, with c real, the continuous spectrum in the bifurcation point extends into the unstable complex half plane. By continuity of the spectrum, all wave trains close to a fold of wave trains are unstable. As we show in [Rademacher & Scheel, 2006], the saddle-node of a homogeneous oscillation may robustly give stable oscillations on one branch of the fold. It is also accompanied by saddle-node bifurcations of wave trains. In this example, the wave trains with non-zero wavenumber undergo a sideband instability before entering the fold point.

3.2 Homogeneous oscillations: $c_{\mathrm{g}} = 0$, $k = 0$

The additional reflection isotropy of homogeneous oscillations enriches the scenario of possible robust instabilities. The symmetry (8) implies that the set of spatial Floquet exponents ν is reflection symmetric with respect to the imaginary axis as well, so that for marginally stable spectrum $d(\mathrm{i}\alpha, \mathrm{i}\ell) = 0$ implies $\partial_\nu d(\mathrm{i}\alpha, \mathrm{i}\ell) = 0$, in particular the coefficients in (10) of odd powers vanish, so $\partial_\nu d(0,0) = 0$, hence $c_{\mathrm{g}} = 0$.

Hopf: We assume (LS) and (S0), and that (S) is violated for a unique pair of double Floquet exponents (up to Floquet multiples) $\lambda = \pm\mathrm{i}\alpha \notin \{0, \mathrm{i}\frac{\omega}{2}\}$, $\nu = \pm\mathrm{i}\ell \neq 0$, where $d = 0$ and for (10) we have

$$\mathrm{Re}(a_2) > 0.$$

Pure Hopf: We assume (LS) and (S0), and that (S) is violated for a unique double pair (up to Floquet multiples) $\lambda = \pm\mathrm{i}\alpha \notin \{0, \mathrm{i}\frac{\omega}{2}\}$, $\nu = 0$, where $d = 0$ and for (10) we have

$$\mathrm{Re}(a_2) > 0.$$

Period-Doubling: We assume (LS) and (S0), and that (S) is violated for a unique double (up to Floquet multiples) $\lambda = \mathrm{i}\frac{\omega}{2}$, $\nu = 0$, where $d = 0$ and for (10) we have

$$a_2 > 0.$$

Sideband instability: We assume (S) and (S0), and that (LS) is violated so that at $\lambda = \nu = 0$ for (10) we have

$$a_2 = 0, \quad a_4 < 0.$$

Fold: We assume (S) and (S0), and that (LS) is violated so that at $\lambda = \nu = 0$

$$d = 0, \quad \partial_\lambda d = 0, \quad \partial_{\lambda\lambda} d \neq 0, \quad \partial_{\nu\nu} d(0,0) \neq 0.$$

Note that the action of the reflection symmetry is trivial on the kernel since $\nu = 0$, so we expect the fold, which is typical for the pure kinetics.

Turing: We assume (S) and (LS), and that (S0) is violated at $\lambda = 0$, and a unique pair $\nu = \pm \mathrm{i}\ell \neq 0$, where $d = 0$ and for (10) we have

$$a_2 > 0.$$

Using that $\lambda(\mathrm{i}\alpha) = \tilde{\lambda}(\alpha^2)$, similar arguments with $\rho = \alpha^2$ as in the case of $k \neq 0$ show that Hopf, Period Doubling and Sideband are robust. Pure Hopf is robust due to the symmetry: the double Floquet exponent $\nu = 0$ cannot be perturbed away from the origin at marginal stability. The Turing instability is robust since we can conclude from the symmetry $d(\lambda, \nu) = d(\lambda, -\nu)$ that

$$d_s(\lambda) := d(\lambda, \mathrm{i}\ell) = d(\lambda, -\mathrm{i}\ell) = \overline{d(\overline{\lambda}, \mathrm{i}\ell)} = \overline{d_s(\overline{\lambda})};$$

hence $d_s : \mathbb{R} \to \mathbb{R}$ and assuming $\partial_\mu d(0; \mu = 0) \neq 0$ gives a robust root. Therefore, the location of marginal instability at $(\lambda, \nu) = (0, \mathrm{i}\ell)$ is fixed in the sense that a perturbation may only change the critical wavenumber ℓ.

The symmetric sideband instabilities occur in the complex Ginzburg-Landau equation at the Benjamin-Feir limit [Cross & Hohenberg, 1993]. Pure Hopf, Period-doubling bifurcations and folds can be detected in the pure kinetics and yield PDE examples with identity or near-identity diffusion matrices (the near-identity diffusion matrix excludes instabilities $\ell \neq 0$ which would not stem from the kinetics alone). Hopf bifurcations can be realized in systems where a Hopf bifurcation is coupled to an oscillatory Turing bifurcation of an equilibrium.

3.3 Turing patterns: $\omega = 0$, $k \neq 0$, reflection symmetry

Again, the reflection symmetry leads to a scenario analogous to the case of the homogeneous oscillations.

Hopf: We assume (LS) and (S0), and that (S) is violated for a unique pair of double Floquet exponents $\lambda = \pm \mathrm{i}\alpha \neq 0$, $\nu = \pm \mathrm{i}\ell \notin \{0, \mathrm{i}\frac{k}{2}\}$, where $d = 0$ and for (10) we have

$$\mathrm{Re}(a_2) > 0.$$

Pure Hopf: We assume (LS) and (S0), and that (S) is violated for a unique double pair (up to Floquet multiples) $\lambda = \pm i\alpha \neq 0$, $\nu = 0$, where $d = 0$ and for (10) we have

$$\mathrm{Re}(a_2) > 0.$$

Spatial Period-Doubling: We assume (LS) and (S0), and that (S) is violated for a unique (up to Floquet multiples) $\lambda = 0$, $\nu = \mathrm{i}\frac{k}{2}$, where $d = 0$ and for (10) we have

$$a_2 > 0.$$

Sideband instability: We assume (S) and (S0), and that (LS) is violated so that at $\lambda = \nu = 0$, where for (10) we have

$$a_2 = 0, \quad a_4 < 0.$$

Fold/Pitchfork: We assume (S) and (S0), and that (LS) is violated at $\lambda = \nu = 0$, where

$$d = 0, \quad \partial_\lambda d = 0, \quad \partial_{\lambda\lambda} d \neq 0, \quad \partial_{\nu\nu} d \neq 0.$$

As for homogeneous oscillations, a fold occurs, if the kernel lies in the reflection symmetric subspace. The pitchfork corresponds to an odd action, such that the bifurcating patterns are not reflection symmetric and typically travel, $\omega \neq 0$.

Turing: We assume (S) and (LS), and that (S0) is violated at $\lambda = 0$, and a unique (up to Floquet multiples) pair $\nu = \pm \mathrm{i}\ell \notin \{0, \mathrm{i}\frac{k}{2}\}$, where $d = 0$ and for (10) we have

$$a_2 > 0.$$

The robustness of all cases follows as in the homogeneous case using symmetry.

A variant of the Hopf bifurcation occurs in the Taylor-Couette experiment, when Taylor vortices destabilize to wavy vortices [Chossat & Iooss, 1994]. The pure Hopf occurs in the Chlorite-Iodite-Malonic-Acid reaction [Castets et al., 1990], where an oscillation destabilizes a Turing pattern. Sideband instabilities are common at onset of convection patterns and well understood in the Ginzburg-Landau equation [Cross & Hohenberg, 1993]. Stationary bifurcations appear to be less commonly observed, although examples can be readily constructed in amplitude equations.

4 Discussion

4.1 Genericity

We did not claim and only provided little evidence that our lists are exhaustive for a given class of PDE. Our non-robustness result, Lemma 3.5, relies on perturbations of the dispersion relation. It is however not clear if reaction-diffusion systems provide a sufficiently large class to realize the type of perturbations which are necessary here. A related difficulty stems from the loss of information in passing to the determinant d: symmetry properties of the matrix are only reflected in some crude form in the determinant. For instance, on the level of the dispersion relation, it seems impossible to decide whether the eigenfunction is even or odd, leading to the distinction of a Fold versus a pitchfork bifurcation in the Turing case. In this case, the determinant is given through

$$d(\lambda, \nu) = \det\left(\Phi^0_\lambda - \mathrm{e}^{2\pi\nu/k}\right),$$

with $\Phi^0_\lambda := \Psi_{\lambda,0}$, and inherits a reversibility property

$$\Phi^0_\lambda = R(\Phi^0_\lambda)^{-1}R, \quad R(u,v)^T = (u,-v)^T,$$

from the reflection symmetry.

4.2 Modulation Equations

We do not attempt to derive or even justify amplitude equations in all the different scenarios. Modulations of the neutral mode in wave trains with $\omega = 0$ are described by a Burgers equation, with viscosity and nonlinear flux given by expansions of the linear and nonlinear dispersion relation, respectively [Doelman et al., 2006]. Justifications of modulation equations in other cases far from homogeneous equilibria do not appear to be available. Formal derivations can be found in various contexts, see for instance [Cross & Hohenberg, 1993, Van Harten, 1995] and the references therein. Turing patterns are described by a nonlinear phase-diffusion equation. In case of a sideband instability, the sign of the viscosity changes and higher-order derivatives appear, leading to Cahn-Hillard type equations for Turing patterns, Kuramoto-Sivashinsky equations for homogeneous oscillations, and singularly perturbed Korteweg-deVries for generic wave trains [Van Harten, 1995]. In case of an additional critical mode, Burgers or phase diffusion equations are coupled to amplitude equation for the instability. For generic wave trains, group velocities of wave trains and additional unstable modes typically differ, leading to inconsistencies in the expansion. Generally, Hopf for generic and pure Hopf for symmetric wave trains is modeled by a single complex Ginzburg-Landau equation and Hopf in the symmetric case by a pair of coupled complex Ginzburg-Landau equations. Turing modes are described by Ginzburg-Landau equations, the period-doubling mode by an Allen-Cahn equation.

4.3 Bifurcations

Instead of trying to understand the nonlinear dynamics of the continuous band of unstable and marginally stable modes via modulation equations, one can focus on solutions which are periodic with a minimal period adapted to the instability. The simplest example here would be to prescribe spatially periodic boundary conditions that accommodate both the wave train and an unstable mode. More precisely, we consider the reaction-diffusion system in the comoving frame $y = x - \frac{\omega}{k}t$ and L-periodic boundary conditions, $L = 2\pi\frac{m}{k}$. Here, $m \in \mathbb{Z}$, so that $\frac{m}{k} = \frac{j}{\ell}$ for some $j \in \mathbb{Z}$, and $0 < \ell \leq k$ is the most unstable wavenumber. The wave train then corresponds to a circle of equilibria, generated by the $SO(2)$ symmetry of spatial translations in periodic boundary conditions. The isotropy of the wave train is $\mathbb{Z}_m$, the cyclic group, or D_m, the dihedral group in the case of reflection symmetric wave trains. Bifurcation equations are skew products, where an equation for the "normal modes", equivariant with respect to the isotropy action [Vanderbauwhede, 1982, Golubitsky et al., 1988], couple to an equation for the motion along the circle of equilibria [Krupa, 1990, Fiedler et al., 1996, Golubitsky & LeBlanc 2000].

This approach is limited in several ways. First, it is impossible to capture the onset when ℓ is non-resonant, a typical phenomenon except for the period-doubling case. Sideband instabilities are particularly difficult to capture in this context. A first partial remedy would be to study instabilities varying the period as an additional external parameter. Next, the bifurcation diagram is difficult to interpret since it contains only stability information within the prescribed period. A remedy for this drawback would be to complement the bifurcation analysis within the class of periodic boundary conditions with a stability analysis for bifurcating wave trains in the class of bounded functions on the real line. This introduces yet another parameter in the bifurcation analysis, namely the modulational wavenumber ℓ, in the stability analysis. As a prototype for such an analysis, we have studied the fold of a homogeneous wave train in [Rademacher & Scheel, 2006].

An alternative view would impose conditions on the temporal behavior rather than the spatial behavior and focus on stationary or temporally periodic solutions in a frame $y = x - ct$. The spatial eigenvalue $i\ell$ of the instability then corresponds to a purely imaginary Floquet exponent of a periodic orbit in the spatial dynamics. Bifurcating spatial patterns include in particular all harmonic and subharmonic resonant bifurcations from periodic orbits and reversible periodic orbits, see e.g. [Ciocci & Vandebauwhede, 2004, Vanderbauwhede, 2000, Knobloch & Vanderbauwhede, 1995] and the references therein. In this spatial dynamics picture, we recover the difficulty of choosing a common spatial period for the wave train and the unstable wavenumber in the form of frequency locking: for subharmonic bifurcations, above onset, resonant bifurcating orbits dominate the dynamics also for non-resonant Floquet multipliers $\frac{\ell}{k} \notin \mathbb{Q}$. While this approach overcomes the obstacle of choosing an appropriate spatial period, it lacks information on non-stationary or non-periodic temporal dynamics. A partial remedy would be to complement this spatial dynamics analysis with a *temporal* stability analysis for the bifurcating wave trains. [Sandstede & Scheel, 2000b, Doelman et al., 1995, Doelman et al., 2006, Mielke, 1997, Haragus et al., 2006]. Spatial dynamics also exhibit a variety of coherent structures beyond periodic solutions, such as heteroclinic and homoclinic connections between periodic orbits, corresponding to phase boundaries between different wave trains, e.g. [Sanstede & Scheel, 2004].

In summary, a complete understanding of spatially and temporally coherent structures in typical bifurcations seems to be a challenging enterprise, certainly beyond the scope of this article.

4.4 First Unstable Wavenumbers

So far, we have concentrated on instabilities of individual wave trains. Since wave trains come in one-parameter families, there typically is a critical curve $\mu(k)$, in the (k, μ)-parameter plane of existing wave trains, such that the wave trains with wave number k destabilize at $\mu = \mu(k)$. We refer to local maxima of this curve as *first instabilities* with corresponding *first unstable wave number*. Since $\mu(k) = \mu(-k)$, this view point justifies the exceptional treatment that we gave to the case $k = 0$, since instabilities at $k = 0$ occur in a robust fashion as first instabilities when $\mu(0) = 0$, $\mu''(0) > 0$. We expect the list of typical bifurcations for first instabilities to be the same as the list for instabilities of wave trains, since the restriction $\mu'(k) = 0$ is compensated for by an additional parameter.

The more global point of view on wave trains as families of solutions rather than individual solutions also leads to a variety of new global questions: where do curves of wave trains terminate, and what are stability properties at those end points. The most elementary end points for paths of periodic orbits are Hopf bifurcations near equilibria and homoclinic and heteroclinic bifurcations. We do not attempt to classify those bifurcations but refer to the extensive literature on pathfollowing for periodic orbits [Fiedler, 1988, Fiedler & Heinze, 1996a, Fiedler & Heinze, 1996b], local bifurcations [Mielke, 1997, Mielke, 2002], and global bifurcations [Sandstede & Scheel, 2001b]. Our contribution here is to possible PDE instabilities on global branches of periodics, *away* from those local and global bifurcations.

4.5 Absolute Instabilities

Instabilities at non-symmetric wave trains are typically convective, that is, $\partial_\nu d \neq 0$, so that the linearization possesses a linear drift term $\lambda = -c_{\rm g}\nu$ as a leading order term. Note that also for first instabilities, we expect this to be the case since the parameter k is needed to match the condition $\mu'(k) = 0$. The effect of a convective instability is that localized perturbations will decay at each fixed point x of physical space, while their overall norm grows [Briggs, 1964, Sandstede & Scheel, 2000a]. In particular, the linearization at wave trains in bounded domains with separated boundary conditions, for instance wave trains generated by a Dirichlet source on the boundary of a large domain, will remain stable beyond the onset of the instability. The linear instability sets in when the *absolute spectrum* of the linearization at the periodic orbit crosses the imaginary axis. Roughly speaking, the absolute spectrum consists of semi-algebraic curves $\lambda(\gamma)$ where $d(\lambda, \nu_1) = d(\lambda, \nu_2) = 0$ and $\nu_1 = \nu_2 + {\rm i}\gamma$, so that real parts of roots ν_j to $d(\lambda, \nu) = 0$ have a certain fixed distribution relative to $\operatorname{Re}\nu_1$; for (1) with $d_j > 0$, $j = 1, .., n$ they should be equi-distributed. In any large bounded domain with separated boundary conditions, the spectrum of the linearization at a wave train is approximated by its absolute spectrum and a finite number of eigenvalues. In particular, eigenvalue clusters accumulate at the semi-algebraic curves, determined by the above condition on the dispersion relation. We refer to [Rademacher et al., 2005] for a theoretical description of absolute spectra of wave trains and a practical guide to the computation of these spectra.

We expect a similar classification as in Table 1 for absolute instabilities. Since absolute spectra terminate in double roots of the dispersion relation, where $\nu_1 = \nu_2$, we find a first list of branch points λ crossing at $\lambda = 0$ or $\lambda = \pm{\rm i}\omega$, with double root $\operatorname{Im}\nu = 0$ or $\operatorname{Im}\nu \neq 0$, leading to four instability mechanisms similar to the classification four spatially homogeneous equilibria. However, we also expect instabilities where curves of absolute spectrum touch the imaginary axis with quadratic tangency; see [Sandstede & Scheel, 2000a, Example 2].

Note that for reflection symmetric wave trains, absolute and essential spectra coincide and the previous classification applies to absolute instabilities as well.

4.6 Concluding Remarks

We laid out a systematic approach to bifurcations from spatio-temporally periodic patterns in extended domains in essentially one-dimensional media. As opposed to classifications based on symmetry only, we allow for arbitrary wavenumber perturbations, but require marginal stability with respect to all wavenumbers at criticality. As expected, spatio-temporal period-doubling bifurcations and sideband instabilities enrich the list of instabilities as compared to bifurcations from homogeneous equilibria. Somewhat surprisingly, the fold is expected to be an uncommon phenomenon as the onset of an instability, because it is preceded by sideband instabilities.

In many of the instabilities that we described, results on dynamics with periodic boundary conditions or for pure traveling wave solutions are known. We expect that one will find many interesting phenomena in the unfoldings of the associated bifurcations when effects of large or unbounded domains are taken into account. We propose to study these effects by complementing the bifurcation analysis in a periodic boundary condition setting with a stability analysis on the entire real line, and give a prototypical example in [Rademacher & Scheel, 2006]. On the other hand, an analysis that imposes *temporally instead of spatially* periodic boundary conditions can provide different insight, in particular when complemented with a temporal stability analysis. Some of the spatially coherent structures have been investigated in the case of stable wave trains [Doelman et al., 2006] (see also [Haragus & Scheel, 2006] for a stability analysis) and in the case of a period-doubling instability [Sandstede & Scheel, 2006], but the field is wide open for most of the other instabilities.

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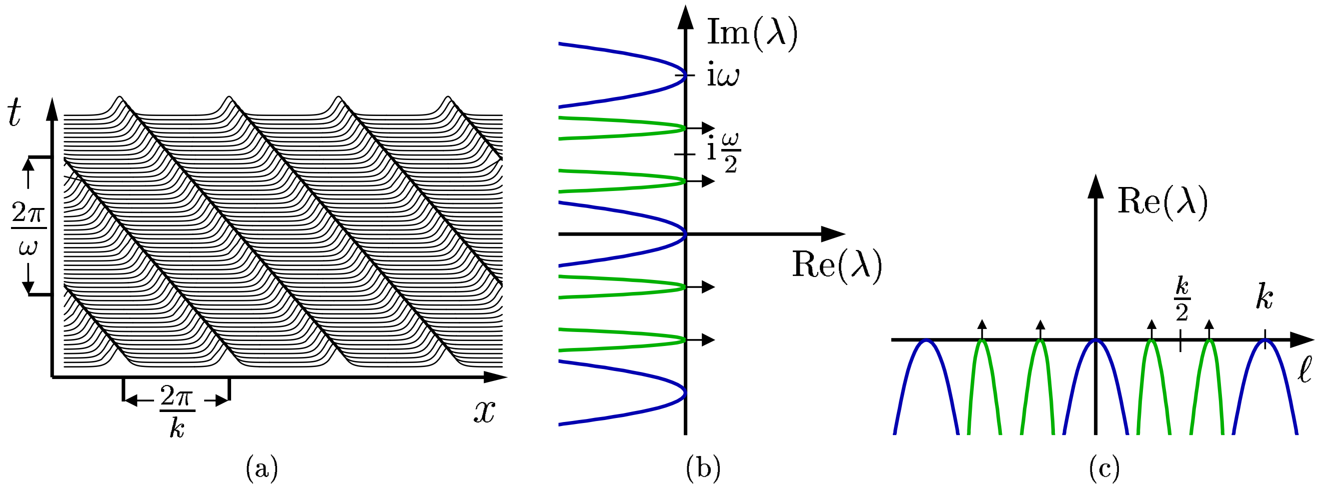

Figure 2: Illustration of the Hopf case of a generic wave train with frequency ω and wavenumber k. (a) sketch of space-time plot, (b) critical spectrum $\lambda \in \mathbb{C}$ in the complex plane, (c) parametrization of the curves $\lambda(\ell)$ by ℓ. Arrows in (b) and (c) indicate the motion as μ moves through the instability.

	Hopf $\alpha \neq 0$ $\ell \neq 0$	Per. dbl. $\alpha = \omega/2$ $\ell = k/2$	Sideband $\alpha = 0$ $\ell = 0$	Pure Hopf $\alpha \neq 0$ $\ell = 0$	Fold $\alpha = 0$ $\ell = 0$	Turing $\alpha = 0$ $\ell \neq 0$
Generic $\omega, k \neq 0$	$\surd$	$\surd$	$\surd$			
Oscillation $\omega \neq 0, k = 0$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Turing $\omega = 0, k \neq 0$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Equilibria $\omega = k = 0$	$\surd$			$\surd$	$\surd$	$\surd$

Table 2: List of robust codimension-one instabilities with respect to a mode $e^{i(\alpha t - \ell x)}$ of a wave train with frequency ω and wavenumber k.